TMUA challenge

Question 1

Let k be a real number, and define $f_k(x) = x^2 - k$ for all real x. It is given that there exist three distinct real numbers a, b, c in arithmetic progression such that

$$\{f_k(a), f_k(b), f_k(c)\} = \{a, b, c\}.$$

Which of the following is the set of all possible values of k?

A. $\left\{\frac{21}{16}\right\}$

B. $\left\{\frac{29}{16}\right\}$

C. $\left\{\frac{21}{16}, \frac{29}{16}\right\}$

D. $\left[\frac{21}{16}, \frac{29}{16}\right]$

E. $\left\{\frac{45}{16}, \frac{53}{16}\right\}$

Question 2

Four lamps are initially off. On each of four turns, a pair of distinct lamps is chosen uniformly at random from the six possible pairs, independently of all previous choices. Both lamps in the chosen pair are switched: an off lamp becomes on, and an on lamp becomes off.

Given that all four lamps are off after the fourth turn, what is the probability that every lamp has been switched at least once?

A. $\frac{1}{4}$

B. $\frac{3}{7}$

C. $\frac{1}{2}$

D. $\frac{15}{28}$

E. $\frac{5}{9}$

Question 3

The vertices A, B, C, D of a square of side length 1 are labelled in order. For a point P strictly between B and C, let ℓ_P be the perpendicular bisector of AP.

There are exactly two positions of P for which ℓ_P divides the square into two regions with areas $\frac{2}{5}$ and $\frac{3}{5}$. Let the two corresponding lines meet at Q, and let X_1 and X_2 be their respective intersections with AB.

What is the area of triangle QX_1X_2?

A. $\dfrac{1}{40}$

B. $\dfrac{\sqrt{5}}{80}$

C. $\dfrac{\sqrt{5}}{40}$

D. $\dfrac{\sqrt{5}}{20}$

E. $\dfrac{3\sqrt{5}}{40}$

Question 4

The function f is defined for all real x by

$$f(x) = x^3 - 3x.$$

For a positive real number h, consider the following statements.

$P(h)$ For every real number x, there exists a real number t with $0 < t \leq h$ such that $f(x+t) > f(x)$.

$Q(h)$ There exists a real number t with $0 < t \leq h$ such that, for every real number x, $f(x+t) > f(x)$.

For exactly which values of h is $P(h)$ true and $Q(h)$ false?

A. There are no such values of h.

B. $0 < h \leq 2\sqrt{3}$.

C. $3 \leq h \leq 2\sqrt{3}$.

D. $3 < h < 2\sqrt{3}$.

E. $3 < h \leq 2\sqrt{3}$.

Question 5

For a real number $C \geq 1$, consider the following statement:

> For every positive real number x, there is an integer n (positive, zero or negative) such that at least one of x, $3x$ and $5x$ lies in the interval
>
> $$[2^n, C\,2^n).$$

What is the smallest value of C for which this statement is true?

A. $\dfrac{6}{5}$

B. $\dfrac{5}{4}$

C. $\dfrac{4}{3}$

D. $\dfrac{3}{2}$

E. $\dfrac{8}{5}$

Question 6

Let $r > 1$. The points A, B and C on the curve $y = x^2$ have x-coordinates 1, r and r^2, respectively.

Let R be the whole region bounded by the straight line segment AC and the arc of the curve from A to C. The area of triangle ABC is exactly half the area of R.

What is the value of r?

A. $\dfrac{3+\sqrt{5}}{2}$.

B. $2+\sqrt{3}$.

C. $3+\sqrt{3}$.

D. $\dfrac{7+3\sqrt{5}}{2}$.

E. $5+2\sqrt{6}$.

Question 7

An angle θ is chosen uniformly at random from $0 \leq \theta < 2\pi$. Thus the probability that θ lies in an interval is the length of that interval divided by 2π. Three points are defined by

$$A = (\cos\theta, \sin\theta), \qquad B = (\cos 2\theta, \sin 2\theta), \qquad C = (\cos 5\theta, \sin 5\theta).$$

What is the probability that these points form an acute triangle whose angle at A is strictly greater than each of its other two angles?

A. $\dfrac{1}{42}$

B. $\dfrac{1}{21}$

C. $\dfrac{1}{7}$

D. $\dfrac{3}{14}$

E. $\dfrac{1}{3}$

Question 8

Choose a positive integer a_1 that is not a perfect square. Starting with a_1, form a sequence by the following rule. If the current term a_n is a perfect square, stop. Otherwise, put

$$a_{n+1} = a_n + 2\left\lfloor \sqrt{a_n} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

How many possible values of a_1 cause the sequence to stop at 3600?

A. 30

B. 39

C. 40

D. 41

E. 59

Question 9

Six real numbers, which need not be distinct, have arithmetic mean 0 and median 1. The arithmetic mean of their squares is 5.

What is the greatest possible value of the largest of the six numbers?

A. $\sqrt{15} - 1$

B. 3

C. 4

D. 5

E. $3\sqrt{3}$

Question 10

For a real number a, the function f is defined by

$$f(x) = \frac{x^2 - 2x + a}{x^2 + 2x + a}$$

where the denominator is nonzero.

Which of the following statements are true for every real value of a?

I. The equation $f(x) = 1$ has exactly one real solution.

II. If $f(x) = -1$ has a real solution, then $f(x) = t$ has a real solution for every real number t.

III. If $f(x) \geq 0$ for every x in its domain, then f is defined for every real x.

A. None of them

B. I only

C. II only

D. III only

E. I and II only

F. I and III only

G. II and III only

H. I, II and III

Question 11

A student attempts to find the number of pairs of positive integers (x, y) such that $x > y$, the highest common factor of x and y is 1, and

$$x^2 - y^2 = 2^5 \times 3^2 \times 7^2.$$

The student's argument is as follows.

I. The integers x and y must both be odd.

II. Put $u = (x - y)/2$ and $v = (x + y)/2$. Then u, v are positive integers, $u < v$, $uv = 2^3 \times 3^2 \times 7^2$, and their highest common factor is 1.

III. Since uv is even and u, v have highest common factor 1, exactly one of u, v is even.

IV. Each of the three prime powers 2^3, 3^2 and 7^2 must lie entirely in u or entirely in v. Before imposing $u < v$, this gives $2^3 = 8$ ordered choices for (u, v).

V. No choice has $u = v$, so exactly four choices have $u < v$.

VI. Conversely, each of these four choices gives positive integers $x = u + v$, $y = v - u$ whose highest common factor is 1.

VII. Therefore there are exactly four pairs (x, y) satisfying the conditions.

Which of the following best describes the argument?

A. The argument is entirely correct.

B. The first error is in line I.

C. The first error is in line II.

D. The first error is in line III.

E. The first error is in line IV.

F. The first error is in line V.

G. The first error is in line VI.

H. The first error is in line VII.

Question 12

The line $y = mx + \frac{3}{2}$, where $m > 0$, meets the graph $y = |x^2 - 1|$ at exactly three distinct points. Find the sum of all possible values of m.

A. 0

B. $\dfrac{3}{2} - \sqrt{2}$

C. $\sqrt{2}$

D. $\dfrac{3}{2}$

E. $\dfrac{3}{2} + \sqrt{2}$

Question 13

Let $f(x) = x^2 + ax + b$, where a and b are real numbers. Consider the statements

P $\quad f(x) \geq 0$ for every real number x.

Q $\quad f(n)f(n+1) \geq \frac{1}{16}$ for every integer n.

Which of the following is true?

A. Q is necessary but not sufficient for P.

B. Q is sufficient but not necessary for P.

C. Q is both necessary and sufficient for P.

D. Q is neither necessary nor sufficient for P.

Question 14

The real numbers x and y satisfy

$$\log_2(1+x) + \log_2(1+y) = 0, \qquad x^3 + y^3 = 4.$$

What is the value of $(x-y)^4$?

A. 1

B. 5

C. 9

D. 16

E. 25

Question 15

The equation

$$6(\sin^3\theta + \cos^3\theta) = 5(\sin\theta + \cos\theta)$$

has n distinct solutions $\theta_1, \ldots, \theta_n$ in $0 \le \theta < 2\pi$. What is the value of

$$\tan^2\theta_1 + \tan^2\theta_2 + \cdots + \tan^2\theta_n\,?$$

A. 10

B. 35

C. 68

D. 70

E. 74

F. 140

Question 16

A sequence is defined by

$$u_1 = a, \qquad u_{n+1} = u_n(2 - u_n) \quad (n \geq 1),$$

where a is real.

Which of the following statements are true for every real value of a?

I. If $u_{50} = u_{52}$, then $u_1 = u_2$.

II. If $u_{50} \geq 0$, then $0 \leq a \leq 2$.

III. If a is rational and u_{50} is an integer, then a is an integer.

A. None of them

B. I only

C. II only

D. III only

E. I and II only

F. I and III only

G. II and III only

H. I, II and III

Question 17

For which real values of a does the equation

$$x^4 - 2ax^2 + 16x + a^2 = 0$$

have at least one real solution?

A. $a \leq -3$

B. $a \geq -3$

C. $a > -3$

D. $a \geq -1$

E. $a \geq 0$

Question 18

Tangents to the circle $x^2 + y^2 = 1$ at distinct points A and B meet at $P = (p, 2)$. The straight line through A and B passes through $(2, 0)$.

Find the area of quadrilateral $OAPB$, where O is the origin.

A. $\dfrac{\sqrt{13}}{4}$

B. $\sqrt{3}$

C. $\dfrac{\sqrt{13}}{2}$

D. $\dfrac{\sqrt{17}}{2}$

E. $\sqrt{13}$

Question 19

Let $S = \{1, 2, 3, 4, 5\}$. A function $f : S \to S$ is chosen uniformly at random from all such functions. It is given that $f(f(x)) = f(x)$ for every $x \in S$, and that exactly two elements of S satisfy $f(x) = x$. What is the probability that $f(1) = f(2)$?

A. $\dfrac{3}{20}$

B. $\dfrac{1}{5}$

C. $\dfrac{3}{10}$

D. $\dfrac{9}{20}$

E. $\dfrac{7}{15}$

F. $\dfrac{1}{2}$

Question 20

Let a, b and c be positive real numbers such that $abc = 1$. Consider the following statements.

P $\quad a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Q $\quad$ For every integer $n \geq 2$,

$$a^n + b^n + c^n \geq \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}.$$

Which of the following is true?

A. P is necessary and sufficient for Q.

B. P is necessary but not sufficient for Q.

C. P is sufficient but not necessary for Q.

D. P is neither necessary nor sufficient for Q.

Question 21

Let m and n be positive integers. In the polynomial

$$(1+x)^m(1-x)^n,$$

the coefficient of x^2 is zero, and the coefficient of x^3 is -8 times the coefficient of x.

What is the degree of the polynomial?

A. 5

B. 15

C. 24

D. 25

E. 49

Question 22

The polynomial f has real coefficients, has degree at most 2, and satisfies

$$\int_0^1 f(x)\,dx = 0, \qquad \int_0^1 (f(x))^2\,dx = 1.$$

What is the greatest possible value of $f(0)f(1)$?

A. -3

B. 0

C. 1

D. $\frac{5}{2}$

E. 5

F. 10

Question 23

The integers a, b and c satisfy

$$a^2 + b^2 + c^2 = (a + b + c)^2.$$

Which of the following statements must be true?

I. abc is divisible by 4.

II. abc is divisible by 8.

III. $(a + b + c)abc \leq 0$.

A. None of them

B. I only

C. II only

D. III only

E. I and II only

F. I and III only

G. II and III only

H. I, II and III

Question 24

For a real number a, the curves

$$y = x^2 - a \qquad \text{and} \qquad x = y^2 - a$$

intersect at the four vertices of a convex quadrilateral of area $\sqrt{21}$.

Find a.

A. -1

B. $\dfrac{2+\sqrt{37}}{8}$

C. $\dfrac{5}{4}$

D. $\dfrac{3}{2}$

E. $\dfrac{1+2\sqrt{22}}{4}$

Question 25

An arithmetic progression (u_n) and a geometric progression (v_n) have real terms and are both nonconstant. They satisfy

$$u_1 = v_1 = 1, \qquad u_2 = v_2, \qquad u_4 = v_4.$$

Which of the following statements are true?

I. There are exactly three positive integers n for which $u_n = v_n$.

II. For all positive integers j, k, if $u_j = v_k$, then $j = k$.

A. Neither I nor II

B. I only

C. II only

D. Both I and II

Question 26

The real number k is such that the equation

$$|\log_2 x| + |\log_2(4 - x)| = k$$

has four distinct real solutions. Their product is P.

Which of the following gives all possible values of P?

A. $\frac{64}{25} < P < 3$

B. $3 < P < 4$

C. $9 < P < \frac{256}{25}$

D. $9 \leq P < \frac{256}{25}$

E. $9 < P \leq \frac{256}{25}$

Question 27

Let f and g be quadratic polynomials with real coefficients.

Consider the following statements.

I. The equations $f(g(x)) = x$ and $g(f(x)) = x$ have the same number of distinct real solutions.

II. The equations $f(g(x)) = f(x)$ and $g(f(x)) = g(x)$ have the same number of distinct real solutions.

Which of these statements must be true for every such pair f and g?

A. Neither I nor II

B. I only

C. II only

D. Both I and II

Question 28

The set $\{1, 2, \ldots, 8\}$ is partitioned at random into four unordered pairs, with every such partition equally likely. Let D be the sum of the positive differences between the two integers in each pair.

Given that $D \leq 8$, what is the probability that 1 and 2 form a pair?

A. $\dfrac{3}{35}$

B. $\dfrac{1}{7}$

C. $\dfrac{1}{3}$

D. $\dfrac{9}{19}$

E. $\dfrac{4}{7}$

F. $\dfrac{5}{7}$

Question 29

The points A and B have coordinates $\left(0, \frac{1}{4}\right)$ and (a, a) respectively, where a is real. As P varies over the parabola $y = x^2$, the minimum value of $PA + PB$ is $\frac{3}{4}$.

What is the sum of the possible values of a?

A. $\dfrac{1-\sqrt{17}}{8}$

B. $\dfrac{5-\sqrt{17}}{8}$

C. $\dfrac{1}{4}$

D. $\dfrac{1}{2}$

E. $\dfrac{3}{4}$

F. $\dfrac{5+\sqrt{17}}{8}$

Question 30

Let a, b, c be real numbers such that $a + b + c = 0$.

Consider the following conditions.

$$\begin{aligned} &\mathrm{P}: \quad a^2 + b^2 + c^2 \geq 6. \\ &\mathrm{Q}: \quad |x-a| + |x-b| + |x-c| \geq |x+1| + |x| + |x-1| \quad \text{for every real } x. \end{aligned}$$

Which of the following is correct?

A. P is necessary and sufficient for Q.

B. P is necessary but not sufficient for Q.

C. P is sufficient but not necessary for Q.

D. P is neither necessary nor sufficient for Q.

Question 31

Extension

In how many orders can the numbers $1, 2, 3, 4, 5, 7$, each used once, be written so that, for each $k = 1, 2, 3, 4, 5$, the mean of the first k numbers is greater than the mean of all six numbers?

A. 112

B. 116

C. 120

D. 124

E. 216

F. 360

Question 32

A convex quadrilateral $ABCD$ is inscribed in a circle, with $AB = 1$ and $CD = 3$. The diagonals AC and BD are perpendicular, and AC/BD is a positive integer.

What is the sum of the distinct possible areas of $ABCD$?

A. $\dfrac{9}{26}$

B. $\dfrac{32}{13}$

C. $\dfrac{84}{13}$

D. $\dfrac{207}{26}$

E. $\dfrac{168}{13}$

Question 33

A horizontal line meets the curve

$$y = x^3 - 3x$$

at three distinct points. The gradients of the tangents at these points, when placed in increasing order, form an arithmetic progression.

What is the product of the three gradients?

A. -54

B. -18

C. -2

D. 27

E. 54

Question 34

The angles α and β satisfy

$$0 < \alpha, \beta < \frac{\pi}{2}, \qquad \alpha + \beta > \frac{\pi}{2}, \qquad \cos\alpha\cos\beta = \frac{1}{4}.$$

What is the complete range of $\tan\alpha + \tan\beta$?

A. $(2, 4)$

B. $(2\sqrt{3}, 4)$

C. $[2\sqrt{3}, 4)$

D. $[2\sqrt{3}, 4]$

E. $(\sqrt{15}, 4)$

Question 35

The real numbers x and y satisfy $0 < x < 1$, $0 < y < 1$ and

$$(\log_2 x)(\log_2 y) = \log_2(x + y).$$

What is the greatest possible value of xy?

A. $2^{1-\sqrt{5}}$

B. $2^{1-\sqrt{3}}$

C. $2^{(1-\sqrt{5})/2}$

D. $2^{1+\sqrt{5}}$

E. There is no greatest value.

Question 36

For $a > 1$, the arithmetic mean and median of the five numbers

$$(x-a)^2, \quad (x-1)^2, \quad x^2, \quad (x+1)^2, \quad (x+a)^2$$

are equal for exactly four distinct real values of x.

What is the complete range of a?

A. $(1, 2)$

B. $\left[\sqrt{\frac{3}{2}}, 2\right)$

C. $\left(\sqrt{\frac{3}{2}}, 2\right)$

D. $\left(\sqrt{\frac{3}{2}}, 2\right]$

E. $\left(\sqrt{\frac{3}{2}}, 2\right) \cup (2, \infty)$

Question 37

Let $a > 0$ and define

$$u_n = \left(1 - \frac{a}{n}\right)^n \qquad (n = 1, 2, 3, \ldots).$$

Which of the following statements are true for every value of a?

I. If $u_3 \geq 1 - a$, then $u_n \geq 1 - a$ for every positive integer n.

II. If $u_4 \geq 1 - a$, then $u_n \geq 1 - a$ for every positive integer n.

III. There are infinitely many positive integers n for which $u_n \geq 1 - a$.

A. None of I, II and III

B. I only

C. II only

D. III only

E. I and II only

F. I and III only

G. II and III only

H. I, II and III