TMUA

COMMUNITY PAPERS

Viluxent Set 1 Paper 1

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: Viluxent

Question 1

Real numbers x, y, z satisfy

$$x + y + z = 6, \qquad xy + yz + zx = 9.$$

Find the number of ordered triples (x, y, z) of real numbers for which xyz attains its maximum possible value.

A 1

B 2

C 3

D 4

E 6

F Infinitely many

Question 2

How many values of $x \in [0, 2\pi]$ satisfy both

$$4\sin^2 x - 4\sin x \cos x + \cos^2 x = 0 \qquad \text{and} \qquad 2\sin^2 x + 3\sin x \cos x - 2\cos^2 x = 0$$

simultaneously?

A 0

B 1

C 2

D 3

E 4

F 6

Question 3

How many real solutions does the equation

$$2^x + 2^{-x} = x^2 + 2$$

have?

A 0

B 1

C 2

D 3

E 4

F Infinitely many

Question 4

Find the number of ordered pairs of positive integers (x, y) with $x > y$ such that

$$x^2 - y^2 = 14400$$

and $x - y$ is a perfect square.

A 2

B 3

C 4

D 5

E 6

F 8

Question 5

Let p be a real number, and let n denote the number of points of intersection of the curves

$$y = |x^3 - p^3| \qquad \text{and} \qquad y = p^3|x - 1|.$$

n **cannot** take which of the following values?

A 0

B 1

C 4

D 3

E 2

Question 6

The circle C has equation

$$(x-5)^2+(y-4)^2=9.$$

The two tangent lines from the origin O to C meet at O, making an angle θ between them. Find the exact value of $\cos\theta$.

A $\frac{7}{41}$

B $\frac{9}{41}$

C $\frac{23}{41}$

D $\frac{32}{41}$

E $\frac{39}{41}$

F $\frac{1}{41}$

Question 7

In triangle ABC, the side lengths a, b, c (opposite A, B, C respectively) are in arithmetic progression, and $B = 60^\circ$. Given that the area of the triangle is $10\sqrt{3}$, find the exact perimeter of the triangle.

A $2\sqrt{10}$

B $3\sqrt{10}$

C $6\sqrt{10}$

D $4\sqrt{30}$

E $12\sqrt{10}$

F $6\sqrt{5}$

Question 8

A differentiable function f is defined for all $x > 0$ and satisfies

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y > 0,$$

together with $f(2) = 1$. Find $f'(8)$.

A $\frac{1}{8}$

B $\frac{\ln 2}{8}$

C $\frac{1}{8 \ln 2}$

D $\frac{3}{8 \ln 2}$

E $\frac{1}{8 \ln 8}$

F $\frac{1}{2 \ln 2}$

Question 9

Let

$$P(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots + 3000x^{2999}.$$

Find the remainder when $P(x)$ is divided by $x^2 - x + 1$.

A $3000x$

B $-3000x$

C $3000x - 1$

D $-3000x + 1$

E $1500x$

F $-1500x - 1$

Question 10

Let $\lfloor y \rfloor$ denote the greatest integer less than or equal to y. Find the value of

$$\int_{50}^{1/3} \log_5 \left(\frac{\lfloor 3x \rfloor}{\lfloor 3x \rfloor + 1} \right) dx.$$

A $-\frac{1}{3} \log_5 150$

B $\frac{1}{3} \log_5 150$

C $\log_5 150$

D $-\log_5 150$

E $\frac{1}{3} \log_5 50$

F $3 \log_5 150$

Question 11

Let $g_0(x) = x$ for $x \in [0, 1)$, and for $n \geq 0$ define

$$g_{n+1}(x) = 2g_n(x) - \lfloor 2g_n(x) \rfloor$$

(that is, $g_{n+1}(x)$ is the fractional part of $2g_n(x)$). For each $n \geq 1$, let

$$u_n = \int_0^1 g_n(x)dx.$$

Find the value of $\sum_{n=1}^{100} u_n$.

A 25

B 50

C 75

D 100

E 5050

F $\frac{1}{2}$

Question 12

For a positive integer n, let $d(n)$ denote the number of digits in the decimal representation of n (so $d(7) = 1$, $d(58) = 2$, $d(100) = 3$, and so on). For how many integers $C > 1$ does

$$\sum_{n=1}^{\infty} C^{-d(n)}$$

evaluate to a positive integer?

A 1

B 2

C 3

D 4

E 5

F 9

Question 13

A particle travels from $A(0,0)$ to a point on the line $3x + 4y = 24$, and then on to $B(5,-4)$.

Find the minimum possible total length of its path.

A $\sqrt{41}$

B $\sqrt{137}$

C $\frac{137}{5}$

D $\frac{29\sqrt{2}}{5}$

E $\frac{2\sqrt{221}}{5}$

F 12

Question 14

Positive integers a and b satisfy

$$(a+b+1)(a-b+1) = 7200$$

and

$$\frac{a+1}{b} + \frac{b}{a+1} = \frac{130}{63}.$$

Find the value of a.

A 74

B 105

C 119

D 134

E 146

F 161

Question 15

Three circles, each of radius 2, are mutually tangent to one another (each pair touches at exactly one point). Find the exact area of the curved triangular region enclosed between the three circles.

A $4\sqrt{3} - 2\pi$

B $2\sqrt{3} - \pi$

C $4\sqrt{3} - \pi$

D $8\sqrt{3} - 4\pi$

E $4\pi - 2\sqrt{3}$

F $4\sqrt{3} - \frac{4\pi}{3}$

Question 16

Let $f(x) = \frac{x}{x-1}$ for $x \neq 1$, and let g be the inverse function of f. How many real solutions does the equation

$$f(x) = g(x)$$

have?

A 0

B 1

C 2

D 3

E Infinitely many

Question 17

Let $f(x) = \frac{3x-1}{x+2}$. How many real solutions does the equation

$$f(f(x)) = x$$

have?

A 0

B 1

C 2

D 3

E 4

Question 18

Let p be a non-zero real constant, and consider the curve

$$y = x^4 - 4px^3 + 6p^2x^2 + 1.$$

For how many values of p does this curve have a point of inflection?

A 0

B 1

C 2

D Every non-zero p works

E Only $p = 1$ works

F Only $p = \pm 1$ work

Question 19

Which of the following is true?

A $5^{4000} > 4^{5000}$

B $5^{4000} < 4^{5000}$

C $5^{4000} = 4^{5000}$

D It is impossible to determine which is larger without a calculator

Question 20

Triangle PQR has $PQ = 5$, $QR = 8$, and $\angle PQR = 60^\circ$. Point M lies on QR with $QM = 3$. An ant starts at P at time $t = 0$ and walks in a straight line directly to M at constant speed, arriving at M at time $t = 6$. At the same instant $t = 0$, a second ant starts at Q and walks directly to R along QR at constant speed, arriving at R at time $t = 8$.

Find the shortest distance between the two ants at any point during the interval while both ants are still moving (i.e. for $0 \le t \le 6$).

A 3

B $\frac{5}{2}$

C $\frac{15\sqrt{3}}{14}$

D 5

E $\frac{45}{28}$

F $\frac{3\sqrt{3}}{2}$