TMUA

COMMUNITY PAPERS

Viluxent Set 1 Paper 2

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: Viluxent

Question 1

Let n be a positive integer, and consider the statements

$$P : n^2 - 1 \text{ is divisible by } 24. \qquad Q : \gcd(n, 6) = 1.$$

Which one of the following is true?

A P is sufficient but not necessary for Q

B P is necessary but not sufficient for Q

C P is necessary and sufficient for Q

D P is neither sufficient nor necessary for Q

Question 2

You have n stacks of coins, each containing n coins weighing n grams each. However, one stack is defective, and every coin in that stack weighs only $n-1$ grams. You are given a scale that reports the exact total weight (in grams) of whatever is placed on it.

What is the minimum number of measurements needed to determine which stack is defective, for every $n \geq 2$?

A 1

B 2

C $\lceil \log_2 n \rceil$

D $n-1$

E n

F It cannot always be determined

Question 3

How many solutions does

$$3\sin^2 x + 5\sin x \cos x - 2\cos^2 x = 0$$

have for $0 \leq x < 2\pi$?

A 2

B 3

C 4

D 5

E 6

F 8

Question 4

Let b and c be real numbers with $c \neq 0$.

P: The equation $x^2 + bx + c = 0$ has two distinct positive real roots.

Q: The equation $cx^2 + bx + 1 = 0$ has two distinct positive real roots.

Which one of the following is true?

A P is necessary but not sufficient for Q

B P is sufficient but not necessary for Q

C P is neither necessary nor sufficient for Q

D P is necessary and sufficient for Q

Question 5

Find

$$\sum_{k=0}^{\infty} \left\lfloor \frac{1 + \sqrt{\frac{800{,}000{,}000{,}000{,}000}{4^k}}}{2} \right\rfloor .$$

A 20,000,000

B 28,284,270

C 28,284,271

D 28,284,272

E 14,142,136

F 40,000,000

G 56,568,542

H 8,000,000

Question 6

Quadratic Chase is a single-player game. To start the game, the player chooses two distinct nonzero integers a and b and writes the equation $x^2 + ax + b = 0$ on a blackboard.

On each turn, if the equation currently written on the blackboard has two distinct nonzero integer solutions $x = u$ and $x = v$, then the player erases the equation and replaces it with one of the equations $x^2+ux+v = 0$ and $x^2 + vx + u = 0$, of their choosing. Otherwise, the game ends.

There is exactly one initial choice of (a, b) for which the game can be played forever. For this pair, find the value of $a + b$.

A -3

B -1

C 0

D 1

E 3

F No such pair exists

Question 7

A sequence is defined by $a_1 = k$, for some real number k, and

$$a_{n+1} = a_n^2 - 2 \quad (n \geq 1).$$

P: The sequence (a_n) is bounded. Q: $-2 \leq k \leq 2$.

Which one of the following is true?

A P is neither necessary nor sufficient for Q

B P is sufficient but not necessary for Q

C P is necessary and sufficient for Q

D P is necessary but not sufficient for Q

Question 8

Find the value of

$$\sum_{k=1}^{1680} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}.$$

A $\frac{1}{41}$

B $\frac{39}{41}$

C $\frac{40}{41}$

D $\frac{41}{42}$

E 1

F $\frac{1680}{1681}$

Question 9

A ray of light travels from $A(1, 3)$ to the y-axis, reflects, travels to the x-axis, reflects again, and then arrives at $B(4, 7)$.

Find the minimum possible total length of the path.

A $5\sqrt{3}$

B $5\sqrt{5}$

C $2\sqrt{35}$

D $10\sqrt{2}$

E $5\sqrt{6}$

F 15

Question 10

Consider the following attempt to solve the equation $|x-3| = 2x+1$:

$$\text{(I)} \quad |x-3| = 2x+1$$

$$\text{(II)} \quad (x-3)^2 = (2x+1)^2$$

$$\text{(III)} \quad x^2 - 6x + 9 = 4x^2 + 4x + 1$$

$$\text{(IV)} \quad 0 = 3x^2 + 10x - 8$$

$$\text{(V)} \quad 0 = (3x-2)(x+4)$$

$$\text{(VI)} \quad x = \frac{2}{3} \text{ or } x = -4$$

The solutions of the original equation are concluded to be $x = \frac{2}{3}$ and $x = -4$.

Which one of the following is true?

A The conclusion is entirely correct.

B The conclusion is not entirely correct; the first error occurs in passing from (I) to (II).

C The conclusion is not entirely correct; the first error occurs in passing from (II) to (III).

D The conclusion is not entirely correct; the first error occurs in passing from (III) to (IV).

E The conclusion is not entirely correct; the first error occurs in passing from (IV) to (V).

F The conclusion is not entirely correct; the first error occurs in passing from (V) to (VI).

Question 11

Let $f(x) = \frac{2x+1}{x-3}$, for $x \neq 3$.

How many real solutions does the equation $f(f(x)) = x$ have?

A 0

B 1

C 2

D 3

E 4

F Infinitely many

Question 12

Two circles of radii 4 and 9 are externally tangent to one another. A common external tangent line touches the two circles at points A and B respectively; O_1 and O_2 are the centres of the two circles.

Find the exact area of the quadrilateral O_1ABO_2.

A 60

B 66

C 72

D 78

E 84

F 90

Question 13

Consider the statement:

"For every real number x, there exists a real number y such that $x + y > 5$ and $xy < 1$."

Which of the following is the correct negation of this statement?

A There exists a real number x such that, for every real number y, $x + y \leq 5$ and $xy \geq 1$.

B For every real number x, there exists a real number y such that $x + y \leq 5$ or $xy \geq 1$.

C There exists a real number x such that, for every real number y, $x + y \leq 5$ or $xy \geq 1$.

D There exists a real number x such that, for every real number y, $x + y \leq 5$ or $xy \geq 1$, and there exists such a y.

E For every real number x, for every real number y, $x + y \leq 5$ or $xy \geq 1$.

F There exists a real number x such that, for every real number y, $x + y < 5$ or $xy > 1$.

Question 14

How many real numbers x satisfy

$$9^x + 25^x = 15^{x+1}?$$

A 0

B 1

C 2

D 3

E 4

F Infinitely many

Question 15

In the expansion of $(1 + x + x^4)^{50}$ as a polynomial in x, find the sum of the coefficients of the odd-power terms (i.e. the terms $x^1, x^3, x^5, ...$).

A $\frac{3^{50}}{2}$

B $\frac{3^{50}-1}{2}$

C $\frac{3^{50}+1}{2}$

D 3^{49}

E $\frac{3^{49}-1}{2}$

F $3^{50} - 1$

Question 16

A sequence satisfies $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n + 1$ for $n \geq 1$.

(I) a_n is odd for every positive integer n.

(II) $a_n + 1$ is a power of 2 for every positive integer n.

(III) $a_{n+3} - a_n$ is even for every positive integer n.

Which of these statements are true?

A (I) only

B (II) only

C (III) only

D (I) and (II) only

E (I) and (III) only

F (II) and (III) only

G all of them

H none of them

Question 17

In triangle ABC, $AB = 6$ and $AC = 10$. The median from A meets BC at its midpoint M, and $AM = 7$. Find the exact area of triangle ABC.

A $12\sqrt{3}$

B $15\sqrt{3}$

C $18\sqrt{3}$

D $20\sqrt{2}$

E 24

F $15\sqrt{5}$

Question 18

A cone is inscribed in a sphere of fixed radius R (the apex and the entire circular base rim lie on the sphere), so that its volume is as large as possible.

Find the ratio of the volume of this cone to the volume of the sphere.

A $\frac{1}{3}$

B $\frac{4}{9}$

C $\frac{8}{27}$

D $\frac{2}{3}$

E $\frac{16}{27}$

F $\frac{32}{81}$

Question 19

Let k be a positive real constant. For each nonzero real number m, define the line $L_m : y = mx + k/m$.

(I) Every line L_m is tangent to the parabola $y^2 = 4kx$.

(II) For all $m > 0$, the lines L_m pass through a common fixed point independent of m.

(III) For every m, the line L_{-m} is the reflection of L_m in the y-axis.

Which of these statements are necessarily true?

A (I) only

B (II) only

C (III) only

D (I) and (II) only

E (I) and (III) only

F (II) and (III) only

G all of them

H none of them

Question 20

Let f be differentiable on $\mathbb{R}$ with $f'(x) > 0$ for all real x. Define $g(x) = \int_0^x f(t)dt$.

$$P : g \text{ is strictly increasing on } \mathbb{R}.$$

$$Q : g \text{ is convex on } \mathbb{R} \text{ (that is, } g''(x) \geq 0 \text{ for all } x).$$

$$R : g(x) > 0 \text{ for all } x > 0.$$

Which of the following is true?

A P only

B Q only

C R only

D P and Q only

E Q and R only

F all of them

G none of them