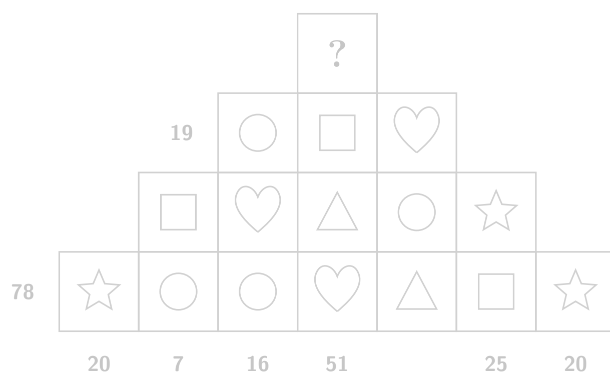
# TMUA EUCLID

## Sample Paper 1

75 MINUTES • NO CALCULATOR




Written by TMUA Euclid

**1.** Let

$$P(x) = x^4 - 4x^3 + 6x^2 - 4x + k.$$

Exactly one real value of $x$ satisfies $P(x) = 0$. What is the value of $k$?

**A** $0$

**B** $1$

**C** $2$

**D** $4$

**E** $8$

**2.** What is the coefficient of $x^9$ in the expansion of

$$(1-x)^7(1+x+x^2)^7?$$

**A** $-84$

**B** $-35$

**C** $-21$

**D** $0$

**E** $35$

**F** $84$

**3.** The triangle with vertices $(1,3)$, $(3,1)$ and $(2,4)$ is reflected in the line $x+2y=2$. Recall that the centroid of a triangle with vertices $(x_1,y_1)$, $(x_2,y_2)$ and $(x_3,y_3)$ is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right).$$

If the centroid of the reflected triangle is $(\alpha,\beta)$, what is the value of

$$15(\alpha-\beta)?$$

**A** $19$

**B** $21$

**C** $22$

**D** $24$

**E** $26$

**4.** For each positive integer $n$, define

$$a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right).$$

Which of the following is true for every positive integer $n$?

**A** $0 < a_n \leq \frac{2}{5}$

**B** $\frac{2}{5} < a_n < \frac{1}{2}$

**C** $\frac{1}{2} < a_n < \frac{3}{5}$

**D** $\frac{3}{5} < a_n < 1$

**E** $a_n > 1$

**5.** Which of the following describes all real solutions of

$$|x^2 - 5| < 2|x|\ ?$$

**A** $|x| < 1 + \sqrt{6}$

**B** $\sqrt{6} - 1 < |x| < \sqrt{6} + 1$

**C** $1 < |x| < \sqrt{6}$

**D** $|x| > \sqrt{6} - 1$

**E** $1 - \sqrt{6} < |x| < 1 + \sqrt{6}$

**6.** The real numbers $a, b, c$ satisfy

$$a + b + c = 6 \qquad \text{and} \qquad ab + bc + ca = 9.$$

What is the greatest possible value of $c$?

**A** $3$

**B** $3 + \sqrt{3}$

**C** $3 + 2\sqrt{3}$

**D** $4$

**E** $6$

**7.** The parabola

$$y = x^2 + px - 3$$

meets the $x$-axis at $Q$ and $R$, and the $y$-axis at $P$. A circle with centre $(-1, -1)$ passes through $P, Q$, and $R$. What is the area of triangle $PQR$?

**A** $4$

**B** $5$

**C** $6$

**D** $7$

**E** $8$

**8.** For how many integers $n$ is

$$\frac{n^2+9}{n+3}$$

a positive integer?

**A** $4$

**B** $5$

**C** $6$

**D** $10$

**E** $12$

**9.** For real $x$, let $\lfloor x \rfloor$ be the greatest integer less than or equal to $x$. What is the value of

$$\sum_{n=1}^{100} \lfloor \sqrt{n} \rfloor ?$$

**A** $575$

**B** $600$

**C** $615$

**D** $625$

**E** $650$

**10.** A sequence $(x_n)$ is defined by

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n}{1 + x_n}.$$

What is the value of

$$\sum_{n=1}^{20} x_n x_{n+1}?$$

**A** $\frac{40}{41}$

**B** $\frac{60}{41}$

**C** $\frac{80}{41}$

**D** $2$

**E** $\frac{40}{21}$

**F** $\frac{82}{41}$

**11.** The trapezium rule, using any positive number of equal-width strips, is applied to

$$\int_0^2 \left(x^4 - 4x^3 + 6x^2\right)\,dx.$$

Which statement is true?

**A** It always gives an overestimate.

**B** It always gives an underestimate.

**C** It is exact only when the number of strips is even.

**D** It is exact for every number of strips.

**E** It gives an overestimate for an odd number of strips and an underestimate for an even number.

**12.** For real $\theta$, define

$$T = \sin\theta + \cos\theta + 2\sin\theta\cos\theta.$$

Let $M$ be the maximum possible value of $T$, and let $m$ be the minimum possible value of $T$. What is $M - m$?

**A** $2 + \sqrt{2}$

**B** $\frac{9}{4} + \sqrt{2}$

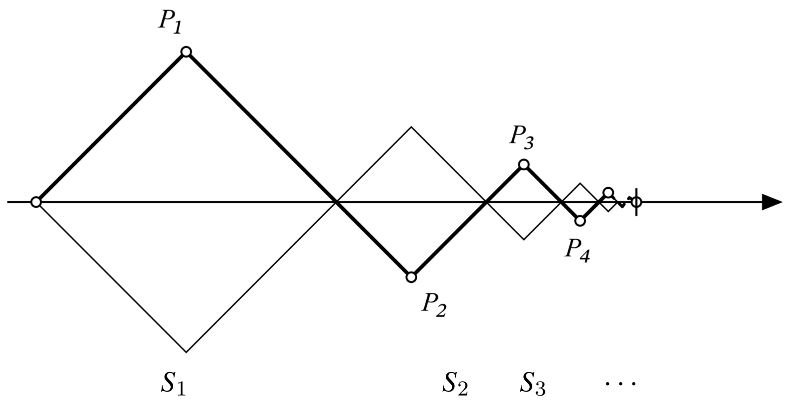
**C** $\frac{5}{2} + \sqrt{2}$

**D** $1 + 2\sqrt{2}$

**E** $\frac{7}{4} + 2\sqrt{2}$

**F** $3$

**13.** A sequence of squares $S_1, S_2, S_3, \ldots$ is arranged as shown. Their horizontal diagonals lie consecutively on the positive $x$-axis. The horizontal diagonal of $S_1$ has length $4$, and each subsequent diagonal has half the length of the preceding one.




For odd $n$, let $P_n$ be the upper vertex of $S_n$. For even $n$, let $P_n$ be the lower vertex of $S_n$. Starting at the origin, a broken line is drawn successively through

$$P_1, P_2, P_3, \ldots$$

The points $P_n$ approach a point on the $x$-axis.
What is the sum of the areas of all the bounded regions between the broken line and the $x$-axis?

**A** $\frac{8}{3}$

**B** $4$

**C** $\frac{14}{3}$

**D** $\frac{16}{3}$

**E** $6$

**F** $8$

**G** $\frac{32}{3}$

**H** $16$

**14.** For which values of $k$ does

$$|x^2 - 1| = kx + 1$$

have exactly three distinct real solutions?

**A** $k = 0$ only

**B** $k = \pm 1$ only

**C** $-1 \leq k \leq 1$

**D** $k \in \{-1, 0, 1\}$

**E** $|k| \geq 1$

**15.** For each integer $k$ satisfying

$$-4 \leq k \leq 4,$$

consider the two curves

$$y = 2^x \qquad \text{and} \qquad y = (-1)^k x + k.$$

Let $N_k$ be the number of points of intersection of these two curves. Find

$$\sum_{k=-4}^{4} N_k.$$

**A** $4$

**B** $5$

**C** $6$

**D** $7$

**E** $8$

**F** $9$

**G** $10$

**H** $12$

**16.** A differentiable function $f : (0, \infty) \to \mathbb{R}$ satisfies

$$f'(x) = -\frac{4}{x^2}.$$

For each $t > 0$, the tangent to the curve $y = f(x)$ at $x = t$ meets the coordinate axes at $P_t$ and $Q_t$. Let $A_t$ be the area of triangle $OP_tQ_t$, where $O$ is the origin. It is given that

$$A_2 = A_5.$$

What is the value of $A_7$?

**A** $0$

**B** $\frac{8}{9}$

**C** $2$

**D** $4$

**E** $6$

**F** $8$

**G** $16$

**H** It cannot be determined

**17.** A monic cubic polynomial $f$ has three distinct real roots. The two bounded regions enclosed by the curve $y = f(x)$ and the $x$-axis have equal areas.

The curve has two stationary points, and the horizontal distance between them is $2$. What is the total area of the two bounded regions?

**A** $\frac{3}{2}$

**B** $2$

**C** $3$

**D** $4$

**E** $\frac{9}{2}$

**F** $6$

**G** $9$

**H** $12$

**18.** A quartic polynomial $P$ has leading coefficient $4$ and satisfies

$$P(x) \geq 0$$

for every real $x$. It has exactly two distinct real roots and also satisfies

$$P(1) = P(-1) \qquad \text{and} \qquad P(2) - P(-2) = 192.$$

The three stationary points of the curve $y = P(x)$ form a triangle. What is the area of this triangle?

**A** $8$

**B** $8\sqrt{2}$

**C** $16$

**D** $16\sqrt{2}$

**E** $24$

**F** $24\sqrt{2}$

**G** $32$

**19.** For a real parameter $p$, consider the two curves

$$C_p : \quad (|x| - 1)^2 + y^2 = (p + 1)^2$$

and

$$D_p : \quad x^2 + (|y| - 1)^2 = (p + 1)^2.$$

The curves intersect at eight distinct points. The four intersection points nearest the origin are the vertices of a square $S$, while the other four are the vertices of a square $T$. It is given that

$$\mathrm{Area}(T) = 9\,\mathrm{Area}(S).$$

Find the value of $p^2 + 2p$.

**A** $-\frac{7}{8}$

**B** $-\frac{5}{8}$

**C** $-\frac{3}{8}$

**D** $-\frac{1}{8}$

**E** $\frac{1}{8}$

**F** $\frac{3}{8}$

**G** $\frac{5}{8}$

**H** $\frac{7}{8}$

**20.** For which values of $k$ does the equation

$$x^6 - 12x^4 - kx^3 - 12x^2 + 1 = 0$$

have exactly four distinct real solutions?

**A** $-22 < k < 22$

**B** $-10\sqrt{5} < k < 10\sqrt{5}$

**C** $-22 \leq k \leq 22$

**D** $-22 < k < 22$, or $k = \pm 10\sqrt{5}$

**E** $22 < |k| < 10\sqrt{5}$