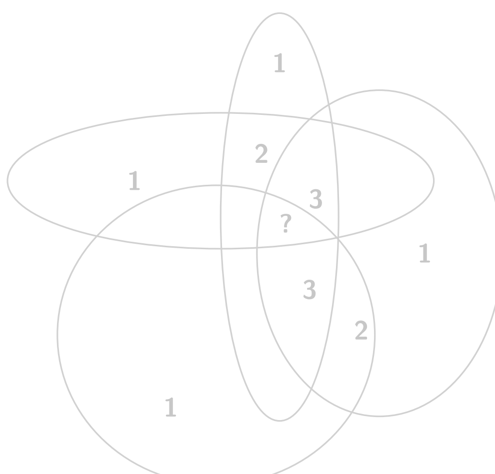
TMUA EUCLID

Sample Paper 2

75 MINUTES • NO CALCULATOR



Written by TMUA Euclid

1. Let $x, y > 0$ and

$$x + y = 1.$$

The following statements may be used to prove

$$\frac{1}{x} + \frac{1}{y} \geq 4.$$

$$\begin{aligned} P &: \quad (x-y)^2 \geq 0, \\ Q &: \quad x^2 + y^2 \geq 2xy, \\ R &: \quad (x+y)^2 \geq 4xy, \\ S &: \quad 1 \geq 4xy, \\ T &: \quad \frac{1}{xy} \geq 4, \\ U &: \quad \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{1}{xy}. \end{aligned}$$

Which ordering gives a valid proof?

A P, Q, R, S, T, U

B P, R, Q, S, U, T

C Q, P, S, R, T, U

D R, P, Q, T, S, U

E U, T, S, R, Q, P

2. A sequence (a_n) satisfies

$$a_{n+2} = a_{n+1} + a_n$$

for every positive integer n.
A student claims:
If a_1 is even, then every term of the sequence is even.
The student writes:

1. a_1 is even.
2. Assume a_k is even.
3. Then $a_{k+1} = a_k + a_{k-1}$, which is the sum of two even numbers.
4. Therefore a_{k+1} is even, completing the induction.

What is the main flaw?

A The recurrence is valid only when n is even.

B The value of a_2 has not been shown to be even.

C An even number plus an even number can be odd.

D Mathematical induction cannot be applied to sequences.

E There is no flaw.

3. A sequence satisfies

$$a_{n+1} = 1 - a_n$$

for every positive integer n. Consider

$$P: \quad \text{The sequence has least period } 2,$$

$$Q: \quad a_1 \neq \frac{1}{2}.$$

Which option correctly describes the relationship between P and Q?

A P is necessary and sufficient for Q

B P is sufficient but not necessary for Q

C P is necessary but not sufficient for Q

D P is neither necessary nor sufficient for Q

4. Consider the statement:
For every real number x, there exists a positive integer n such that

$$x < n \leq x + 2.$$

Which is its logical negation?

A For every real x, every positive integer n satisfies $n \leq x$ or $n > x + 2$.

B There exists a real x and a positive integer n such that $n \leq x$ or $n > x + 2$.

C There exists a real x such that every positive integer n satisfies $n \leq x$ or $n > x+2$.

D For every real x, there exists a positive integer n such that $n \leq x$ or $n > x + 2$.

E There exists a positive integer n such that every real x satisfies $n \leq x$ or $n > x+2$.

5. A function f is continuous on $[0, 1]$. Consider

$$P: \quad f(c) = 0 \text{ for some } c \in (0, 1),$$

$$Q: \quad f(0)f(1) < 0.$$

Which option correctly describes the relationship between P and Q?

A P is necessary and sufficient for Q

B P is sufficient but not necessary for Q

C P is necessary but not sufficient for Q

D P is neither necessary nor sufficient for Q

6. A student calculates

$$n^2 + n + 11$$

for $n = 0, 1, 2, 3$ and obtains four prime numbers. The student conjectures that the expression is prime for every integer $n \geq 0$.
Which statements are true?

I. $n = 10$ is a counterexample.

II. Every positive integer n that leaves remainder 0 or 10 when divided by 11 is a counterexample.

III. Every counterexample must leave remainder 0 or 10 when divided by 11.

A None

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

7. A real sequence (a_n) satisfies

$$a_{n+2} = a_{n+1} - a_n$$

for every positive integer n. Which statements must be true?

I. $a_{n+3} = -a_n$ for every positive integer n.

II. $a_{n+6} = a_n$ for every positive integer n.

III. $a_n + a_{n+1} + \cdots + a_{n+5} = 0$ for every positive integer n.

A none

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

8. Let n be a positive integer. Consider

$$P: \quad 24 \mid n^2,$$

$$Q: \quad 6 \mid n.$$

Which option correctly describes the relationship between P and Q?

A P is necessary and sufficient for Q

B P is sufficient but not necessary for Q

C P is necessary but not sufficient for Q

D P is neither necessary nor sufficient for Q

9. In triangle ABC, let M be the midpoint of BC. Define

$$P: \quad \angle BAC = 90^\circ,$$

$$Q: \quad AM = \frac{1}{2}BC,$$

$$R: \quad MA = MB = MC.$$

Which statements are true?

I. P if and only if Q.

II. Q if and only if R.

III. R only if P.

A I only

B II only

C III only

D I and II only

E I and III only

F II and III only

G I, II and III

H None of them

10. The positive real numbers

$$a < b < c$$

are the side lengths of a non-degenerate triangle. Define

$$x = b + c - a, \qquad y = c + a - b, \qquad z = a + b - c.$$

Which statements must be true?

I. $z < y < x$.

II. x, y, z are necessarily the side lengths of a non-degenerate triangle.

III. $x + y + z = a + b + c$.

A none

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11. For how many integers n with

$$1 \le n \le 500$$

is

$$n^2 + 3n + 2$$

divisible by 35?

A 28

B 42

C 56

D 57

E 70

12. The following lines can be arranged to prove that there are no positive integers a and b satisfying

$$a^2 = 2b^2.$$

P. Suppose that such pairs exist, and choose one for which a is as small as possible.

Q. Since a^2 is even, a is even; write $a = 2c$.

R. Substitution gives $b^2 = 2c^2$.

S. Hence b is even; write $b = 2d$.

T. Substitution now gives $c^2 = 2d^2$.

U. But $c = a/2 < a$, so (c, d) is a smaller positive-integer solution. This is a contradiction.

Which is the correct order?

A P, Q, R, S, T, U

B P, Q, S, R, T, U

C Q, P, R, S, U, T

D P, R, Q, S, T, U

E P, Q, R, T, S, U

13. Let f be twice differentiable on $\mathbb{R}$. Define

$$P: \quad f \text{ has at least four distinct real roots,}$$

$$Q: \quad f' \text{ has at least three distinct real roots,}$$

$$R: \quad f'' \text{ has at least two distinct real roots.}$$

Which statements are true?

I. P only if Q.

II. Q only if R.

III. R only if P.

A I only

B II only

C III only

D I and II only

E I and III only

F II and III only

G I, II and III

H None of them

14. A differentiable function f satisfies

$$f'(x) = x^2 - 4$$

for every real x, and $f(0) = 0$.
Which statements must be true?

I. The equation $f(x) = k$ has three distinct real solutions if and only if

$$|k| < \frac{16}{3}.$$

II. Whenever the three solutions are α, β, γ,

$$\alpha^2 + \beta^2 + \gamma^2 = 24.$$

III. If

$$0 < k < \frac{16}{3},$$

then the middle of the three solutions is positive.

A None

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

15. For each $n = 1, 2, \ldots, 100$, the statement S_n is:
At least n of the integers m with $1 \leq m \leq 100$ have $m^2 - 1$ divisible by 24.
Exactly how many of the statements $S_1, S_2, \ldots, S_{100}$ are true?

A 26

B 32

C 33

D 34

E 50

F 67

16. A differentiable function $f : \mathbb{R} \to \mathbb{R}$ is defined as *wobbly* if there exists a real number x such that

$$f'(x) = 0.$$

The function f is known not to be wobbly and satisfies

$$f(-1) < 0 < f(1).$$

Which of the following functions must be wobbly?

I. $g(x) = (f(x))^2$,

II. $h(x) = f(x^2 - 2x) + f(x^2 + 2x)$,

III. $k(x) = f(x^3 + x)$.

A none

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

17. Define

$$f(x) = x^2 - 2x.$$

For a real number k, the equation

$$f(f(x)) = k$$

has exactly four distinct real solutions. Which of the following statements must be true?

I. $-1 < k < 3$.

II. The product of the four solutions is k.

III. Exactly two of the four solutions lie strictly between 0 and 2 if and only if $k > 0$.

A none

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

18. A differentiable function f satisfies

$$f'(x) = (x^2 - 1)(x^2 - 4)$$

for every real x. Which of the following statements must be true?

I. $f(x) + f(-x) = 2f(0)$ for every real x.

II. The curve $y = f(x)$ has exactly two local minimum points.

III. $f(-1) < f(-2) < f(2) < f(1)$.

A none

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

19. For real k, consider

$$|x^2 - 4x + 3| = k|x - 2| + \frac{3}{2}.$$

For which values of k does the equation have exactly six distinct real solutions?

A $k < -\frac{3}{2}$

B $-\frac{3}{2} < k < -\sqrt{2}$

C $-\sqrt{2} < k < 0$

D $-\frac{3}{2} \leq k < -\sqrt{2}$

E $-\frac{3}{2} < k \leq -\sqrt{2}$

F $k > -\sqrt{2}$

20. For a real number a, consider the equation

$$x^2 + 2xy + ay^2 = 1.$$

Define

P : For every real x, there exists a real y satisfying the equation,

Q : For every real y, there exists a real x satisfying the equation,

$$R : \quad a > 0.$$

Which statements are true?

I. P if and only if both Q and R are true.

II. Q only if at least one of P and not R is true.

III. R only if at least one of P and not Q is true.

A I only

B II only

C III only

D I and II only

E I and III only

F II and III only

G I, II and III

H None of them