TMUA Mock Paper 2

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

PAPER 2

75 MINUTES

By sed :)

For questions or corrections, please contact me at @just.sed on Discord. Good luck!
Answer key and a QR code to the full solutions are on the last 2 pages.

1. The equation

$$x^2 + 1 = c(x + 1)$$

has real solutions for x for precisely:

A. all values of c;

B. $-2\sqrt{2} - 2 \leqslant c \leqslant 2\sqrt{2} - 2$;

C. $c \leqslant -2\sqrt{2} - 2$ **or** $c \geqslant 2\sqrt{2} - 2$;

D. $c = 0$ **or** $c = 1$;

E. no values of c.

2. The number of solutions to the equation

$$\log_2(x^3 + 2x^2 + 2x + 1) = 2\log_2(x + 1)$$

is:

A. 0;

B. 1;

C. 2;

D. 3;

E. 4;

F. 5.

3. Consider the statements

$$P : \int_0^a \cos x \; dx = 0;$$

$$Q : \int_0^a \sin x \; dx = 0.$$

It follows that:

A. P is **necessary but not sufficient** for Q;

B. P is **sufficient but not necessary** for Q;

C. P is **necessary and sufficient** for Q;

D. P is **neither sufficient nor necessary** for Q.

4. A polynomial function has M maxima, m minima, and R distinct real roots.

Which of the following equations is **not** possible?

A. $M + m = 6$;

B. $M + m + R = 1$;

C. $M + m + 1 = R$;

D. $2M + 3 = R$;

E. $2m + 1 = R$.

5. Let P, Q be two *if-then* statements. **If** P is **sufficient** for Q, **then**:

A. the contrapositive of Q is **sufficient** for the contrapositive of P;

B. the negation of P is **insufficient** for the negation of Q;

C. the contrapositive of P is **necessary** for the negation of Q;

D. if Q is **sufficient** for P, **then** the negation of P is **necessary and sufficient** for the negation of Q;

E. if P is **unnecessary** for Q, **then** Q is **sufficient** for P.

6. Let $-1 \leqslant k \leqslant 1$ and $a > 0$.

The precise range of a for which **there exists** a k such that

$$\sin(ax) = k$$

has 2 solutions for $0 \leqslant x \leqslant 2\pi$ is:

A. $\frac{1}{4} < a < \frac{7}{4}$;

B. $\frac{1}{4} < a < 1$;

C. $\frac{1}{4} < a \leqslant \frac{11}{4}$;

D. $\frac{1}{4} < a < \frac{11}{4}$;

E. $\frac{1}{2} \leqslant a < \frac{11}{4}$;

F. $\frac{1}{2} < a < \frac{11}{4}$.

7. The side lengths of a right triangle are in arithmetic sequence.

Which of the following statements **must** be true?

- **I.** The perimeter of the triangle is longer than twice the length of the hypotenuse.
- **II.** The difference between the largest and smallest angles in the triangle is greater than $60°$.
- **III.** **If** all side lengths are integers, **then** the area of the triangle is divisible by 6.

A. None of the above;

B. **I** only;

C. **II** only;

D. **III** only;

E. **I**, **II** only;

F. **I**, **III** only;

G. **II**, **III** only;

H. **I**, **II**, and **III**.

8. A quadrilateral $ABCD$ satisfies $\angle ABC = \angle BCD$.

Which of the following (additional) information, taken independently, is **sufficient** to determine each angle of $ABCD$?

I. $ABCD$ is a cyclic quadrilateral.

II. $\angle CDA = 90°$.

III. The values of $\dfrac{AC}{AB}$ and $\dfrac{BD}{CD}$.

A. None of the above;

B. **I** only;

C. **II** only;

D. **III** only;

E. **I**, **II** only;

F. **I**, **III** only;

G. **II**, **III** only;

H. **I**, **II**, and **III**.

9. A degree 4 polynomial $Q(x)$ satisfies $Q(0) = 0$ and is tangent to $y = x$ at two different points.

It follows that the possible numbers of distinct real roots of $Q(x) = 0$ are:

A. $1, 2, 3, 4$

B. $1, 2, 3$

C. $1, 2, 4$

D. $1, 3, 4$

E. $2, 3, 4$

F. $2, 3$

G. $2, 4$

H. $3, 4$

10. Consider the following statement:

> A function $f(x)$ is *uniformly continuous* if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all u and v, if $|u - v| < \delta$, then $|f(u) - f(v)| < \varepsilon$.

It follows that $f(x)$ is **not** uniformly continuous if and only if:

A. For all $\varepsilon > 0$, there exist $\delta > 0$ and u, v such that $|u - v| \geqslant \delta$ and $|f(u) - f(v)| \geqslant \varepsilon$.

B. For all $\varepsilon > 0$, there exist $\delta > 0$ and u, v such that $|u - v| < \delta$ and $|f(u) - f(v)| \geqslant \varepsilon$.

C. For all $\varepsilon, \delta > 0$, there exist u, v such that $|u - v| < \delta$ and $|f(u) - f(v)| < \varepsilon$.

D. For all $\varepsilon, \delta > 0$ and u, v, $|u - v| \geqslant \delta$ and $|f(u) - f(v)| \geqslant \varepsilon$.

E. There exists $\varepsilon > 0$ where for all $\delta > 0$, there exist u, v such that $|u - v| \geqslant \delta$ and $|f(u) - f(v)| \geqslant \varepsilon$.

F. There exists $\varepsilon > 0$ where for all $\delta > 0$, there exist u, v such that $|u - v| < \delta$ and $|f(u) - f(v)| \geqslant \varepsilon$.

G. There exist $\varepsilon, \delta > 0$ and u, v such that $|u - v| \geqslant \delta$ and $|f(u) - f(v)| < \varepsilon$.

H. There exist $\varepsilon, \delta > 0$ such that for all u and v, $|u - v| \geqslant \delta$ and $|f(u) - f(v)| \geqslant \varepsilon$.

11. A student attempts to prove the statement

For all primes $p > 3$, $p^3 - 3p^2$ has at least 12 factors.

Their proof is as follows.

$p^3 - 3p^2 = p^2(p-3)$. **(I)**

As $p > 3$, p is odd, therefore $p - 3 = 2a$ for some integer $a \neq 0$, therefore $p^2(p-3) = p^2 \cdot 2 \cdot a$. **(II)**

If $a = p$, then $p = -3$ which is impossible, so $a \neq p$. **(III)**

If N has prime factorisation $p_1^{a_1} \cdot p_2^{a_2} \cdots p_n^{a_n}$, then each factor of N has the form $p_1^{b_1} \cdot p_2^{b_2} \cdots p_n^{b_n}$, where $0 \leqslant b_i \leqslant a_i$ (for $1 \leqslant i \leqslant n$). **(IV)**

If a is prime, then $p^2 \cdot 2^1 \cdot a^1$ has $(2+1)(1+1)(1+1) = 12$ factors. **(V)**

If a is composite, then $p^2 \cdot 2^1 \cdot a = p^2 \cdot 2^1 \cdot q^1 \cdot b$ for some prime q and integer $b > 1$, which has more than 12 factors. **(VI)**

Which of the following best describes the proof?

A. The proof is completely correct.

B. The first error occurs at line **(I).**

C. The first error occurs at line **(II).**

D. The first error occurs at line **(III).**

E. The first error occurs at line **(IV).**

F. The first error occurs at line **(V).**

G. The first error occurs at line **(VI).**

12. Consider the n digit number $B = b_1b_2 \dots b_n$, where each of b_i $(1 \leqslant i \leqslant n)$ equals either 0 or 1.

A student wishes to deduce each digit b_i, thereby deducing B. They are allowed to query the digit sum of **any** 3 distinct digits, i.e. $(b_p + b_q + b_r)$ for $p \neq q \neq r$.

Which of the following must be true?

- **I.** **If** $n = 4$, **then** there exists a strategy to deduce B by querying 2 three-digit sums.
- **II.** **If** $n = 3k$ for some integer k, **then** there exists a B where querying $k + 1$ digit sums **may** be sufficient to deduce B.
- **III.** **If** $n = 10$, **then** there exists a strategy to deduce B by querying 8 three-digit sums.

A. None of the above;

B. **I** only;

C. **II** only;

D. **III** only;

E. **I**, **II** only;

F. **I**, **III** only;

G. **II**, **III** only;

H. **I**, **II**, and **III**.

13. Which of the following is a valid counterexample to the following statement?

> The difference between the squares of 2 primes has at most 4 factors.

A. 18

B. 21

C. 42

D. $129^2 - 2^2$

E. $257^2 - 179^2$

F. None of the above.

14. An *irrational number* is a number that **cannot** be expressed as $\frac{p}{q}$, where p, q are integers. $\sqrt{2}$ and $\log_2 3$ are irrational numbers.

Let x, y, z be nonzero numbers such that the expression

$$x \log_2(3) + y \log_2(6) + z \log_2(12)$$

is an integer.

How many values among x, y, z can be irrational?

A. $0, 1, 2, 3$;

B. $0, 1, 2$;

C. $0, 1, 3$;

D. $0, 2, 3$;

E. $1, 2, 3$;

F. $1, 2$;

G. $1, 3$;

H. $2, 3$.

15. Let a, b, c be positive integers. Then the equation

$$a^b = 2a^c = 4b^c$$

A. has infinitely many solutions for each of a, b and c;

B. specifies one of a, b and c uniquely, and has infinitely many solutions for the other two;

C. specifies two of a, b and c uniquely, and has infinitely many solutions for the other one;

D. specifies a, b and c uniquely;

E. has no solutions.

16. Consider the sequence

- $a_1 = k$;
- $a_n = -\dfrac{1}{2}a_{n-1} + 3$.

for some number k.

Which of the following must be true?

I. **If** there exists $n > 1$ where $a_n < -2$, **then** $k > 10$.

II. **If** $k > 1$, **then** for all $n \geqslant 1$, $|a_n| \leqslant |a_1|$.

III. For all $n \geqslant 1$, $|a_{n+1} - 2| \leqslant |a_n - 2|$.

A. None of the above;

B. **I** only;

C. **II** only;

D. **III** only;

E. **I**, **II** only;

F. **I**, **III** only;

G. **II**, **III** only;

H. **I**, **II**, and **III**.

17. The function

$$f(x) = A\sin x + B\sin^3 x + 1$$

has unknown constants A, B.

A student wishes to deduce the values of A and B, thereby deducing $f(x)$. For any chosen value of x, they are given the value of $|f(x)|$.

The student is given the values of $|f(x)|$ only after choosing all desired inputs for x.

How many pairs of values $(x, |f(x)|)$ are necessary to **guarantee** that the student can deduce $f(x)$?

A. 1;

B. 2;

C. 3;

D. 4;

E. 5;

F. 6;

G. no number of such pairs guarantees the deduction of $f(x)$.

18. Let k be an integer. Approximating

$$\int_{(k-\frac{1}{4})\pi}^{(k+\frac{1}{4})\pi} \frac{1}{\cos x}\, dx$$

using the trapezium rule with n strips results in an underestimation.

Which of the following must be true?

I. Approximating $\displaystyle\int_{(2k-\frac{1}{4})\pi}^{(2k+\frac{1}{4})\pi} \frac{1}{\cos x}\, dx$ using the trapezium rule with n strips results in an overestimation.

II. $\displaystyle\int_{(k-\frac{1}{4})\pi}^{(k+\frac{1}{4})\pi} \frac{1}{\cos^3 x}\, dx > \int_{(k-\frac{1}{4})\pi}^{(k+\frac{1}{4})\pi} \frac{1}{\cos x}\, dx.$

III. $\displaystyle -\frac{\pi}{\sqrt{2}} < \int_{(k-\frac{1}{4})\pi}^{(k+\frac{1}{4})\pi} \frac{1}{\cos x}\, dx < 0.$

A. None of the above;

B. **I** only;

C. **II** only;

D. **III** only;

E. **I**, **II** only;

F. **I**, **III** only;

G. **II**, **III** only;

H. **I**, **II**, and **III**.

19. x, y, z are numbers satisfying

$$|x| + |y| + z < |x| + y + |z| < x + |y| + |z|.$$

Consider the statements:

P: $z^2 > y^2$;

Q: **if** $|x| > |z|$, **then** $2y < x + z$.

It follows that:

A. both statements **must** be true;

B. one statement **must** be true, one statement **may** be true;

C. both statements **may** be true;

D. one statement **must** be true, one statement is **never** true;

E. one statement **may** be true, one statement is **never** true;

F. both statements are **never** true.

20. Consider a set of distinct integers S.

An element N in S is S-*good* if and only if there exist 2 other elements in S that have a mean of N.

An element N in S is S-*unique* if and only if there exist **exactly** 2 other elements in S that have a mean of N.

Let G, U denote the numbers of S-good, S-unique elements respectively.

Which of the following must be true?

I. For all $n \geqslant 3$, there exists a set S of size n where $U = n - 2$.

II. **If** $U = G$, **then** there exist two elements x, y in S where

$$|x - y| \geqslant \frac{U(U+1)}{2}.$$

III. For all integers $g \geqslant 0$, there exists a set S where $U = 0$ **and** $G = g$.

A. None of the above;

B. **I** only;

C. **II** only;

D. **III** only;

E. **I**, **II** only;

F. **I**, **III** only;

G. **II**, **III** only;

H. **I**, **II**, and **III**.

ANSWER KEY

Question	Answer
1	C
2	B
3	A
4	D
5	D
6	D
7	F
8	A
9	E
10	F
11	F
12	G
13	E
14	A
15	E
16	D
17	D
18	F
19	A
20	F

Full Solutions