TMUA Challenge Paper 2 Tmua-p2-challenge-2

20 questions. Recreated from the publicly available paper.

Question 1

Which of the following is a counterexample to the claim that $n^2 + n + 41$ is prime for every positive integer n?

A.$n = 1$

B.$n = 10$

C.$n = 20$

D.$n = 30$

E.$n = 40$

F.$n = 39$

G.$n = 50$

H.there is none

Question 2

Consider the statement: if a and b are both rational then $a + b$ is rational. Which of the following is its contrapositive?

A.If $a + b$ is rational then a and b are both rational.

B.If a and b are not both rational then $a + b$ is irrational.

C.If $a + b$ is irrational then at least one of a and b is irrational.

D.If $a + b$ is irrational then neither a nor b is rational.

E.If a or b is irrational then $a + b$ is irrational.

F.If a and b are both irrational then $a + b$ is irrational.

G.If $a + b$ is rational then at least one of a and b is rational.

H.If at least one of a and b is irrational then $a + b$ is not rational.

Question 3

For the quadratic $ax^2 + bx + c$ with $a \neq 0$ and a, b, c real, the condition $ac < 0$ is

A.necessary but not sufficient for two distinct real roots.

B.necessary and sufficient for two distinct real roots.

C.neither necessary nor sufficient for two distinct real roots.

D.sufficient for two distinct real roots only when $b = 0$.

E.necessary but not sufficient for two distinct real roots of opposite sign.

F.sufficient but not necessary for two distinct real roots.

G.equivalent to the discriminant being positive.

H.necessary and sufficient for two distinct real roots of the same sign.

Question 4

Consider these three statements about real numbers.

P: for every x there exists y such that $y^2 = x$.
Q: there exists x such that $xy = 0$ for every y.
R: for all x and y, if $xy = 0$ then $x = 0$ or $y = 0$.

Which are true?

A.none of them

B.P only

C.Q only

D.R only

E.P and Q

F.P and R

G.Q and R

H.P, Q and R

Question 5

The numbers $1, 2, 3, \ldots, 2026$ are written on a board. Repeatedly, two of the numbers, a and b, are rubbed out and $|a-b|$ is written in their place. After 2025 such steps a single number is left. Which of the following must be true of it?

A.It is 0.

B.It is 1.

C.It is odd.

D.It is even.

E.It is a multiple of 1013.

F.It is at least 1013.

G.It is prime.

H.None of these must be true.

Question 6

The following argument is offered, where a and b are non-zero real numbers.

(1) $a = b$

(2) $a^2 = ab$

(3) $a^2 - b^2 = ab - b^2$

(4) $(a - b)(a + b) = b(a - b)$

(5) $a + b = b$

(6) $2b = b$, so $2 = 1$

At which step does the argument first fail?

A.step (2)

B.step (3)

C.step (4)

D.step (5)

E.step (6)

F.step (1)

G.more than one step fails

H.it does not fail

Question 7

Which statement best justifies the claim that $n^3 - n$ is divisible by 6 for every integer n?

A. n^3 and n always have the same parity.

B. 6 divides n^3 for every integer n.

C. $n^3 - n = n(n^2 - 1)$, and $n^2 - 1$ is always even.

D. The result holds for $n = 1, 2, 3$ and therefore for all n.

E. $n^3 \equiv n \pmod 6$ because 6 has no repeated prime factor.

F. $n^3 - n = (n-1)n(n+1)$, and a product of three integers is always divisible by 6.

G. $n^3 - n$ is even, and every even number is divisible by 6.

H. $n^3 - n = (n-1)n(n+1)$ is a product of three consecutive integers, so one of them is a multiple of 3 and at least one is even.

Question 8

Consider the claim: if 3 divides $a^2 + b^2$ then 3 divides a and 3 divides b, where a and b are integers. Which of the following is correct?

A.True: a square is congruent to 0 or 1 modulo 3, so a sum of two squares is divisible by 3 only when both squares are.

B.True, but only when a and b are both positive.

C.False, as $a = 1$, $b = 2$ shows.

D.False, as $a = 3$, $b = 6$ shows.

E.True, and it follows at once from the fact that 3 is prime, with no further argument needed.

F.False, as $a = 1$, $b = 5$ shows.

G.True, because $a^2 + b^2 = (a + b)^2 - 2ab$ and 3 must therefore divide $2ab$.

H.False, as $a = 2$, $b = 4$ shows.

Question 9

A sequence (a_n) and a number L are given. Consider the statement

$$\text{for every } \varepsilon > 0 \text{ there exists } N \text{ such that } |a_n - L| < \varepsilon \text{ for all } n > N.$$

Which of the following is its negation?

A.For every $\varepsilon > 0$ there exists N such that $|a_n - L| \geq \varepsilon$ for all $n > N$.

B.There exists $\varepsilon > 0$ such that for every N there exists $n > N$ with $|a_n - L| \geq \varepsilon$.

C.There exist $\varepsilon > 0$ and N such that $|a_n - L| \geq \varepsilon$ for all $n > N$.

D.For every $\varepsilon > 0$ and every N there exists $n > N$ with $|a_n - L| \geq \varepsilon$.

E.There exists $\varepsilon > 0$ such that $|a_n - L| \geq \varepsilon$ for every n.

F.For every N there exist $\varepsilon > 0$ and $n > N$ with $|a_n - L| \geq \varepsilon$.

G.There exists N such that for every $\varepsilon > 0$ there exists $n > N$ with $|a_n - L| \geq \varepsilon$.

H.For every $\varepsilon > 0$ there exists N such that $|a_n - L| > \varepsilon$ for some $n > N$.

Question 10

Which of the following establishes that a 10×10 board cannot be tiled by 1×4 pieces, each of which covers four squares in a row or in a column?

A.100 is not divisible by 4, so no tiling exists.

B.Colour the board like a chessboard. Each piece covers two squares of each colour, and the two colours occur unequally often.

C.No piece can be placed in a corner of the board.

D.The board has an even number of squares, so any tiling would use an odd number of pieces.

E.Nothing establishes it: a tiling does exist.

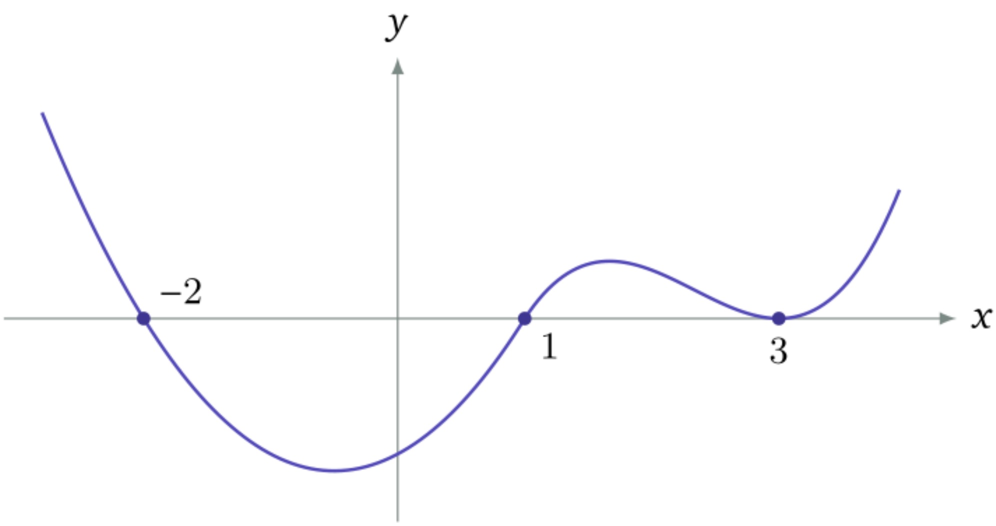
F.Colour the square in row i and column j with $i + j$ modulo 4. Each piece covers all four colours once, so a tiling would need 25 squares of each colour; in fact they occur 25, 26, 25 and 24 times.

G.Colour the columns in the repeating pattern $1, 2, 3, 4, 1, 2, \ldots$ Each piece covers all four colours, and the colours occur 30, 30, 20 and 20 times.

H.A tiling would use 25 pieces, and 25 is odd.

Question 11

The diagram shows a sketch of $y = f'(x)$, the derivative of a function f defined for all real x. The curve meets the x-axis at $x = -2$ and $x = 1$, and touches it at $x = 3$.



Which statement about f is true?

A. f has a local minimum at $x = -2$ and a local maximum at $x = 1$.

B. f has exactly two stationary points.

C. f has a local maximum at $x = 1$ and a local minimum at $x = -2$.

D. f has a local maximum at $x = -2$, a local minimum at $x = 1$, and a stationary point of inflection at $x = 3$.

E. f is decreasing for $x > 3$.

F. f has a local maximum at $x = 3$.

G. f has a local minimum at $x = 3$.

H. f is increasing for $-2 < x < 1$.

Question 12

Five points are placed inside a square of side 2. Which argument shows that some two of them are a distance at most $\sqrt{2}$ apart?

A. Divide the square into four squares of side 1; two of the points lie in the same one, and its diagonal is $\sqrt{2}$.

B. The five points cannot all lie at corners of the square.

C. The average of the ten distances between the points is less than $\sqrt{2}$.

D. Divide the square into five regions; each point lies in one of them.

E. No argument works: the four corners together with the centre is a counterexample.

F. The square has diagonal $2\sqrt{2}$, and half of $2\sqrt{2}$ is $\sqrt{2}$.

G. Divide the square into two rectangles; three of the points lie in the same one.

H. Any five points in the plane include two at distance at most $\sqrt{2}$.

Question 13

On an island there are 13 red, 15 green and 17 blue chameleons. Whenever two chameleons of different colours meet, both change to the third colour; nothing else changes a chameleon's colour. Which of the following states can be reached?

A.All 45 red.

B.All 45 green.

C.All 45 blue.

D.15 of each colour.

E.22 red, 22 green and 1 blue.

F.Any one of ”all red”, ”all green” and ”all blue”.

G.”All green”, but neither of the other two single-colour states.

H.None of the states listed above.

Question 14

The inequality $x^2 + bx + c > 0$ holds for every real x if and only if

A. $b^2 > 4c$

B. $b^2 < 4c$

C. $c > 0$

D. $b^2 \leq 4c$

E. $b^2 < 4c$ and $b > 0$

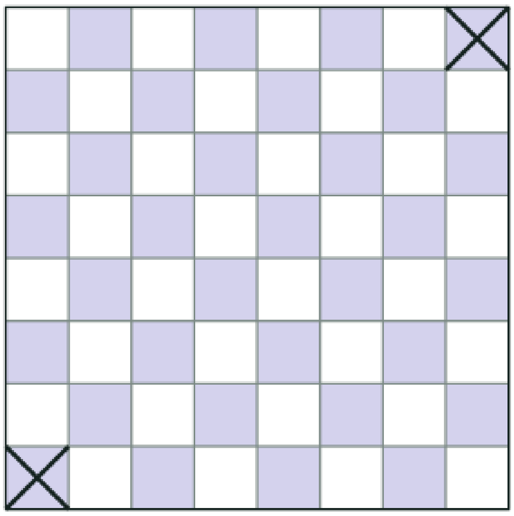
F. $c > 0$ and $b > 0$

G. $b^2 < 4c$ or $c > 0$

H. $b^2 \geq 4c$

Question 15

Two squares at opposite ends of a diagonal are removed from an 8×8 board, as shown. Can the remaining 62 squares be covered exactly by 31 dominoes, each covering two squares that share an edge?



A. Yes, and an explicit covering can be written down.

B. Yes, because 62 is even.

C. No, because 62 is not divisible by 4.

D. No, because no domino can be placed in a corner.

E. Yes, because $62 = 2 \times 31$.

F. No, because an odd number of rows is left uncovered.

G. No: the two removed squares are the same colour, so 32 squares of one colour and 30 of the other remain, while every domino covers one of each.

H. It depends on which pair of opposite corners is removed.

Question 16

Let S be a set of ten distinct positive integers, each less than 100. Which of the following must be true?

A. S contains two elements whose sum is 100.

B. S contains three elements in arithmetic progression.

C. S contains two elements whose difference is 10.

D. The elements of S have sum at least 100.

E. S has two disjoint non-empty subsets with the same sum.

F. S contains two elements, one of which divides the other.

G. S contains a multiple of 10.

H. The largest element of S is at least 55.

Question 17

Consider the statement for all real x, if $x > 1$ then $x^2 > x$. Which of the following is correct?

A.The statement and its converse are both true.

B.The statement is false, as $x = \frac{1}{2}$ shows.

C.The converse is true and the contrapositive is false.

D.The statement and its contrapositive are true, but the converse is false, as $x = -1$ shows.

E.The statement, its converse and its contrapositive are all false.

F.The converse and the contrapositive are both true.

G.The statement is true and its contrapositive is false.

H.The statement is true, and its converse is false, as $x = 2$ shows.

Question 18

How many of the ten digits can occur as the final digit of a perfect square?

A.4

B.5

C.6

D.7

E.8

F.9

G.10

H.3

Question 19

For positive real a and b, which single fact is enough to prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$?

A.$(a-b)^2 \geq 0$

B.$(a+b)^2 \geq 0$

C.$a^2 + b^2 \geq 0$

D.$ab > 0$

E.$\frac{a}{b} > 0$ and $\frac{b}{a} > 0$

F.$a > 0$ and $b > 0$

G.$a^2 + b^2 > ab$

H.$(a-b)^2 > 0$

Question 20

Consider the statement: for every integer n such that $n^2 + n$ is odd, n is a multiple of 7. Which of the following is correct?

A.It is false, as $n = 3$ shows.

B.It is true, vacuously: $n^2 + n = n(n + 1)$ is always even, so there is no such n.

C.It is false, as $n = 1$ shows.

D.It is true, and every such n happens to be a multiple of 7.

E.It is neither true nor false, because no such n exists.

F.It is false, as $n = 7$ shows.

G.It is true, but only because 7 is prime.

H.It cannot be decided without knowing which integers n are allowed.

Answer key

1. Answer: E
2. Answer: C
3. Answer: F
4. Answer: G
5. Answer: C
6. Answer: D
7. Answer: H
8. Answer: A
9. Answer: B
10. Answer: F
11. Answer: D
12. Answer: A
13. Answer: H
14. Answer: B
15. Answer: G
16. Answer: E
17. Answer: D
18. Answer: C
19. Answer: A
20. Answer: B