JZ Mock Set A Paper 1 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

Last updated: 5 September 2026 at 23:13

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Question 1

Find the finite area enclosed between the curve $y = |x^2 - 1|$ and the line $y = 3$.

A $\frac{13}{3}$

B 4

C 8

D $\frac{25}{3}$

E $\frac{23}{3}$

F $\frac{11}{3}$

Solution 1

Answer: C

Note the curve and line meet at $x = 2, -2$, and the setup is symmetric about the y-axis, so we find the area from $x = 0$ to $x = 2$ and double it.

It is easier to calculate area bounded below the curve with x-axis then subtract it from the bounding rectangle.

For $0 < x < 1$, the curve $y = |x^2 - 1| = 1 - x^2$, thus area is $\int_0^1 1 - x^2 dx = 2/3$.

For $1 < x < 2$, the curve $y = |x^2 - 1| = x^2 - 1$, thus area is $\int_1^2 1x^2 - 1dx = 4/3$.

Then $2/3 + 4/3 = 2$, and rectangle area is $2 \times 3 = 6$, therefore final answer is $2 \times (6 - 2) = 8$.

Question 2

The equation $2x^2 + px + 24 = 0$ has two real roots in the ratio $1:3$. Find the value of p^2.

A 256

B 128

C 512

D 64

E 192

Solution 2

Answer: A

Let the two roots be k and $3k$.

$$\begin{aligned} 2x^2 + px + 24 &= 0 \\ x^2 + \frac{p}{2}x + 12 &= 0 \\ (x-k)(x-3k) &= 0 \qquad \text{(roots are } k \text{ and } 3k) \\ x^2 - 4kx + 3k^2 &= 0 \end{aligned}$$

Comparing coefficients:

$$3k^2 = 12 \qquad \Leftrightarrow \qquad k = \pm 2,$$

and

$$\frac{p}{2} = -4k \qquad \Rightarrow \qquad p^2 = 64k^2 = 64 \cdot 4 = 256.$$

Question 3

Find the minimum value of $|x-1|+|x-2|+|x-3|+|x-4|$ over all real x.

A 2

B 3

C 4

D 5

E 6

F 8

G 10

Solution 3

Answer: C

One useful intepretation of $|x-1|$ is as the distance of x from 1. Therefore similarly, The expression $|x-1|+|x-2|+|x-3|+|x-4|$ is the total distance from x to the four points $1, 2, 3, 4$. It is smallest for any x between the two middle points, i.e. $2 \leq x \leq 3$, where

$$(x-1)+(x-2)+(3-x)+(4-x)=4,$$

a constant. The minimum value is therefore 4.

Question 4

Find the sum of all real values of z for which there exist positive real numbers x and y satisfying the simultaneous equations

$$\log_2(x^2yz) = 3, \qquad \log_2(xz) = 1, \qquad (\log_2 y)(\log_2 z) = 2.$$

A 3

B $\frac{9}{4}$

C $\frac{9}{2}$

D 7

E 8

F 9

G $\frac{63}{4}$

Solution 4

Answer: B

Let $X = \log_2 x$, $Y = \log_2 y$, $Z = \log_2 z$. The three equations become

$$2X + Y + Z = 3, \qquad X + Z = 1, \qquad YZ = 2.$$

From the second equation, $X = 1 - Z$. Substituting into the first gives

$$2(1 - Z) + Y + Z = 3 \quad \text{or} \quad Y = 1 + Z.$$

Sub into the third equation: $(1 + Z)Z = 2 \Leftrightarrow (Z + 2)(Z - 1) = 0$ and $Z \in \{-2, 1\}$, these gives $z = \frac{1}{4}, 2$, and so sum of these is $\frac{9}{4}$.

Question 5

Find the area of the region enclosed between the curves $y = 2|x-1|$ and $y = 6 - |x+2|$.

A $\dfrac{11}{2}$

B $\dfrac{20}{3}$

C $\dfrac{17}{3}$

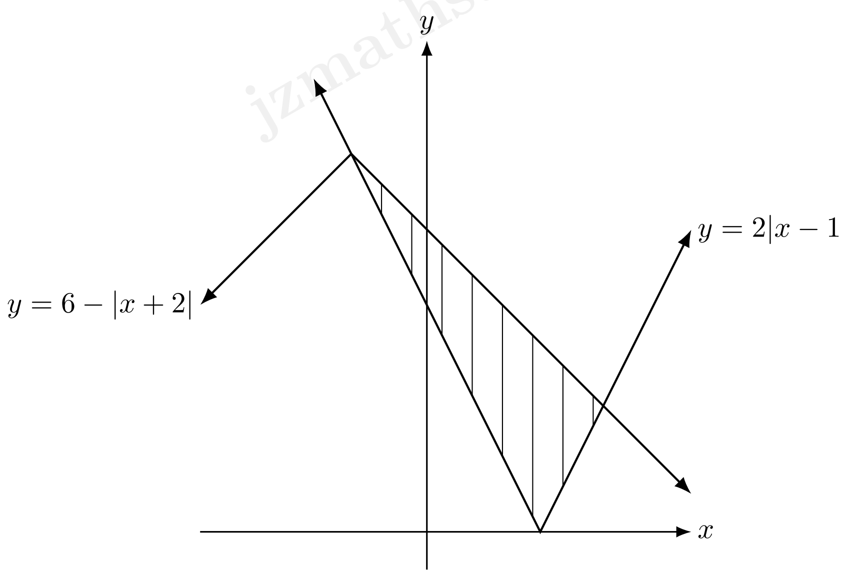
D 6

E $\dfrac{13}{2}$

Solution 5

Answer: D

Start by making a sketch of both modulus V-shaped graphs.



Note that we should also find the two intersections. It turns out it is the negative gradient arm of $y = 6 - |x+2|$ that intercepts both arms of $y = 2|x-1|$. Thus the equations to solve are:

$$6 - (x+2) = 2(x-1) \quad \text{and} \quad 6 - (x+2) = 2(1-x),$$

which gives $x = 2$ and $x = -2$, where $y = 2$ and $y = 6$.

To find the enclosed area, we can just do trapizum subtract two triangles that area under the graph of $y = 2|x-1|$ and the $x-$axis.

Left Triangle area $= \frac{1}{2} \times 6 \times 3 = 9$, right triangle area $= \frac{1}{2} \times 1 \times 2 = 1$ and trapezium area $= \frac{1}{2} \times 4 \times (2+6) = 16$.

Therefore the bounded area is $16 - 9 - 1 = 6$.

Question 6

Given
$$g(x) = \frac{3}{4}\left(\sin\left(\frac{3}{2}x + \frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3} - \frac{3}{2}x\right) + 2\right),$$
find the product of the maximum and minimum value of
$$4g(x) - (g(x))^2 - 3.$$

A -3

B 0

C 1

D 3

E 2

F -1

G 6

Solution 6

Answer: A

If you ever see a scary looking function such as $g(x)$, there is most likly a **catch**, and in this case, there **is** one!

Notice that the arguments of the sin and cos in $g(x)$ adds
$$\left(\frac{3}{2}x + \frac{\pi}{6}\right) + \left(\frac{\pi}{3} - \frac{3}{2}x\right)$$
exactly to $\pi/2$, therefore they are identical. (Using the identity: $\sin x = \cos(\pi/2 - x)$). Therefore, for example:
$$g(x) = \frac{3}{4}\left(2 \cdot \sin\left(\frac{3}{2}x + \frac{\pi}{6}\right) + 2\right).$$
Now, the only thing we need to work out about $g(x)$ is its range, which is now $\{0 \leq g(x) \leq 3\}$, since it is just a scaled and translated single sin function.

Lastly, let $X = g(x)$, $4g(x) - (g(x))^2 - 3 = 4X - X^2 - 3 = 1 - (X-2)^2$. Sketch this out, restrict $0 \leq X \leq 3$ to see the maximum is 1 and minimum is -3, therefore the product is -3.

Question 7

Find the product of all distinct values of p for which

$$(x-p)(x^2+px+p)=0$$

is satisfied by exactly two distinct values of x?

A 0

B -2

C $\frac{7}{2}$

D -4

E $-\frac{7}{2}$

F 2

Solution 7

Answer: B

Clearly $x=p$ is always a solution, thus for the cubic to have 2 distinct solutions, we must have exactly 1 doubly repeated root and a single root. There are two ways this could occur:

(i) $x=p$ is the doubly repeated root.

(ii) The doubly repeated root is not p, and comes from (x^2+px+p).

While examining both cases, we need to be careful to check we don't end up with a single triply repeated root.

case (i): So p must be a root of (x^2+px+p), therefore $p^2+p^2+p=0$, and so $p=0$ or $p=-\frac{1}{2}$. If $p=0$, then the overall equation becomes $x^3=0$, and has only one triply repeated root, so $p\neq 0$. However, we can easily check $p=-\frac{1}{2}$ works as (x^2+px+p) is not a square in this case.

case (ii): For (x^2+px+p) to have a single doubly repeated root, we need its b^2-4ac to be zero. This gives $p^2-4p=0$, so $p=0$ or $p=4$. We already know $p=0$ does not work, and $p=4$ makes (x^2+px+p) into a square, and its root is necessarily not p. Since this only happens when $p=-\frac{1}{2}$. Thus it is definitely not a triply repeated root.

Therefore the product of these two values is $-\frac{1}{2}\times 4=-2$.

Question 8

Let a, b and c be positive integers satisfying

$$18^{a+c}6^{a+b+5} = 12^{b-2}18^5 6^{c+2}.$$

Find the value of $a+b+c$.

A 5

B 6

C 7

D 8

E 9

F 10

Solution 8

Answer: C

Write each base as a product of powers of 2 and 3:

$$6 = 2 \cdot 3, \qquad 12 = 2^2 \cdot 3, \qquad 18 = 2 \cdot 3^2.$$

The left-hand side is

$$2^{(a+c)+(a+b+5)}3^{2(a+c)+(a+b+5)} = 2^{2a+b+c+5}3^{3a+b+2c+5}.$$

The right-hand side is

$$2^{2(b-2)+5+(c+2)}3^{(b-2)+10+(c+2)} = 2^{2b+c+3}3^{b+c+10}.$$

Equating the powers of 2 gives

$$2a+b+c+5 = 2b+c+3,$$

so

$$b = 2a+2.$$

Equating the powers of 3 gives

$$3a + b + 2c + 5 = b + c + 10,$$

so

$$c = 5 - 3a.$$

Therefore

$$a + b + c = a + (2a + 2) + (5 - 3a) = 7.$$

Question 9

The circles C_1 and C_2 have equations

$$x^2 + y^2 - 8y - 64 = 0 \quad \text{and} \quad x^2 + y^2 - 6x - 11 = 0.$$

A **common external tangent** is a line which touches both circles, with the two circles lying on the same side of the line. The two common external tangents to C_1 and C_2 meet at P. One of these common external tangents has equation $y = 2x - 16$. Let A be the centre of C_1 and let B be the centre of C_2. Find the value of AP.

A 12

B 10

C 15

D $3\sqrt{5}$

E $4\sqrt{5}$

Solution 9

Answer: B

You maybe tempted to find the equation of the other tangent, then find P. While that could be done, but it is a tricky calculation, and not the easiest way to find AP. Intead, AP can be found purely geometrically, without knowledge of any of the tangent equations!

C_1: $x^2 + (y-4)^2 = 80$. So C_1 has centre $A = (0, 4)$ and radius $4\sqrt{5}$.

C_2: $(x-3)^2 + y^2 = 20$. So C_2 has centre $B = (3, 0)$ and radius $2\sqrt{5}$.

Let the contact points of one of the tangents with the circles C_1 and C_2 be X and Y respectively. Then observer that triangles AXP and BYP are similar, because they have the same angles.

Now let's first find AB by Pythagorus: $AB = \sqrt{(3-0)^2 + (0-4)^2} = 5$, and let $BP = x$.

By similar triangles:

$$\frac{AX}{AY} = \frac{AP}{BP} \quad \Leftrightarrow \quad \frac{4\sqrt{5}}{2\sqrt{5}} = \frac{5+x}{x} \quad \Leftrightarrow \quad x = 5.$$

Hence $BP = 5$, so $AP = AB + BP = 5 + 5 = 10$.

Question 10

The function

$$f(x) = \frac{1}{4}x^{4/3} + \sqrt[3]{x} + \frac{3}{\sqrt[3]{x^2}}$$

is defined for all real $x \neq 0$. In this question, f is said to be increasing on those parts of its domain where $f'(x) \geq 0$. What is the complete set of values of x for which f is increasing?

A $-3 \leq x < 0,\ x \geq 2$

B $x \leq -3,\ 0 < x \leq 2$

C $x \geq 2$

D $-3 < x < 0,\ x > 2$

E $-3 \leq x \leq 2,\ x \neq 0$

F $-3 \leq x \leq 0,\ x \geq 2$

Solution 10

Answer: A

Write $f(x) = \frac{1}{4}x^{4/3} + x^{1/3} + 3x^{-2/3}$. Differentiating term by term:

$$f'(x) = \frac{1}{3}x^{1/3} + \frac{1}{3}x^{-2/3} - 2x^{-5/3}.$$

Factor out $\frac{1}{3}x^{-5/3}$ (the lowest power):

$$f'(x) = \frac{1}{3}x^{-5/3}\big(x^2 + x - 6\big) = \frac{1}{3}x^{-5/3}(x+3)(x-2).$$

The function is increasing where $f'(x) \geq 0$, but this is one rather daunting looking inequality! Can we divide out $x^{-5/3}$? No, because even though it cannot be zero, as $x \neq 0$, it can still be both positive (if $x > 0$) or negative (if $x < 0$).

A good way to determine the overall sign of $f'(x)$ in this case, is by considering the values of x for which $f'(x)$ changes sign. For example consider what happens around $x = 2$, the value of the other two factors: $x^{-5/3}(x+3)$, around $x = 2$, is not close to zero, and do not vary much around $x = 2$, so we can treat them as 'constant' for purpose of sign change around $x = 2$. But $x - 2$ changes sign on either side of $x = 2$.

Similarly we may deduce that $f'(x)$ changes signs at $x = -3, 0$ and 2. Note that the sign change at 0 is due to $x^{-5/3}$, as mentioned before, it is positive for $x > 0$, negative for $x < 0$.

With these sign changes established, we now merely need to note that for large positive values of x, $f'(x)$ is clearly positive, therefore we can immediately deduce the intervals on which $f'(x) \geq 0$ by sign change as:

$$-3 \leq x < 0, \ 2 \leq x.$$

Note that we need to becareful to include end points of -3 and 2, while exclude $x = 0$ as f is not defined at 0.

Question 11

In the expansion of $(a+bx)^7$, the coefficient of x^5 plus twice the coefficient of x^4 is the same as five times the coefficient of x^3. Given that a and b are positive integers, find the smallest possible value of ab.

A 3

B 5

C 6

D 10

E 15

F 16

G 9

Solution 11

Answer: E

The relevant binomial coefficients in $(a+bx)^7$ are

$$\binom{7}{3}a^4b^3 = 35a^4b^3, \quad \binom{7}{4}a^3b^4 = 35a^3b^4, \quad \binom{7}{5}a^2b^5 = 21a^2b^5$$

for x^3, x^4, x^5 respectively. Apply the given conditions we get:

$$21a^2b^5 + 2(35a^3b^4) = 5(35a^4b^3),$$

which simplifies to:

$$3b^2 + 10ab - 25a^2 = 0,$$

which factorises as $(3b-5a)(b+5a) = 0$. Since $a, b > 0$ the second factor is positive and so cannot be zero, so $3b = 5a$. The smallest possible ab is clearly when $a = 3$ and $b = 5$, and $ab = 15$.

Question 12

The sum to infinity of a geometric sequence (u_n) is $\sqrt{3}$, and the sum to infinity of the sequence obtained by squaring each term of (u_n) is $\sqrt{6}$.

Define a new sequence (v_n) by

$$v_n = (-1)^{n+1} u_n$$

for all integers $n \geq 1$.

What is the sum to infinity of (v_n)?

A $\sqrt{3}$

B $\dfrac{2\sqrt{2}}{3}$

C $\sqrt{2}$

D infinity

E $\dfrac{\sqrt{3}}{2}$

F $\dfrac{\sqrt{2}}{2}$

Solution 12

Answer: C

Let the original geometric sequence be the usual:

$$a,\ ar,\ ar^2, \ldots$$

Since its sum to infinity is $\sqrt{3}$,

$$\frac{a}{1-r} = \sqrt{3}.$$

The sequence obtained by squaring each term is therefore

$$a^2,\ a^2r^2,\ a^2r^4, \ldots$$

so its sum to infinity is

$$\frac{a^2}{1-r^2} = \sqrt{6}.$$

From this point, you could either substitute and solve the simultaneous equations, however I hope $\sqrt{3}$ and $\sqrt{6}$ may discourage you from this path! Instead, notice a useful factorisation:

$$\frac{a^2}{1-r^2} = \frac{a}{1-r} \cdot \frac{a}{1+r} = \sqrt{6}.$$

Therefore

$$\frac{a}{1-r} \cdot \frac{a}{1+r} = \frac{a}{1+r} \cdot \sqrt{3} = \sqrt{6},$$

and

$$\frac{a}{1+r} = \sqrt{2}.$$

Next, write out v_n:

$$a,\ -ar,\ ar^2,\ -ar^3, \ldots$$

whose sum to infinity is

$$\frac{a}{1+r}.$$

Therefore this is immediately equal to $\sqrt{2}$.

Note that it can be shown that the values of a and r are:

$$a = 6\sqrt{2} - 4\sqrt{3}, \qquad r = 5 - 2\sqrt{6},$$

they are not required to answer the question.

Question 13

Find the set of values of a for which the equation

$$x^4 - 4ax^3 + 4a^2x^2 - a = 0$$

has exactly 4 distinct real solutions.

A $a > 1$

B $|a| > 2$

C $a > 2$

D $a > \frac{1}{2}$

E $a > \frac{3}{2}$

Solution 13

Answer: A

Method 1

Rewrite the equation as

$$x^2(x-2a)^2 - a = 0,$$

hence

$$x^2(x-2a)^2 = a.$$

The left-hand side is always non-negative, and $a = 0$ leads to 1 solution, therefore $a > 0$ for there to be 4 real solutions.

Take square roots:

$$x(x-2a) = \pm\sqrt{a},$$

results in two quadratic equations: $x(x-2a) = \sqrt{a}$ and $x(x-2a) = -\sqrt{a}$.

Now a good way to proceed, is to consider the graph if $y = x(x-2a)$, which immediately reveals:

(i) The two equations cannot share any roots unless $a = 0$, and we have $a > 0$ therefore this cannot be the case.

(ii) Since $a > 0$, $x(x-2a) = \sqrt{a}$ will defnitely have 2 distinct roots.

(iii) For $x(x-2a) = -\sqrt{a}$ to have 2 distinct roots, the minimum turning point of $x(x-2a)$ which conveniently is $(a, -a^2)$ by symmetry must be below $y = -\sqrt{a}$.

Therefore, the only requirement now becomes

$$-a^2 < -\sqrt{a} \quad \text{or} \quad a^2 > \sqrt{a},$$

since both sides are positive, we can square both sides to obtain another equivalent inequality:

$$a^4 > a \Leftrightarrow a^3 > 1 \Leftrightarrow a > 1.$$

Method 2

Find the three turning points of $y = x^4 - 4ax^3 + 4a^2x^2 - a$, then use the fact it has 4 distinct real roots if and only if the turning points has y-coordinates with signs $-, +, -$.

Question 14

The following sequence of transformations is applied, in the order listed, to the curve $y = 2x^2 - 3x + 1$.

(1) translation by $\binom{-1}{4}$;

(2) reflection in the line $y = 2$;

(3) stretch parallel to the x-axis with scale factor $\frac{1}{2}$.

What is the equation of the resulting curve?

A $y = -8x^2 - 2x$

B $y = -8x^2 + 2x$

C $y = -2x^2 - x$

D $y = -\frac{1}{2}x^2 - x$

E $y = -\frac{1}{2}x^2 + x$

F $y = -8x^2 - 4x$

G $y = 8x^2 + 2x$

Solution 14

Answer: A

Let $f(x) = 2x^2 - 3x + 1$.

(1) Translation by $\binom{-1}{4}$, in terms of transformations, is $f(x) \longrightarrow f(x+1) + 4$. So

$$y = 2(x+1)^2 - 3(x+1) + 1 + 4 = 2x^2 + x + 4.$$

(2) Reflection in the line $y = 2$, in terms of transformations, is $f(x) \longrightarrow 4 - f(x)$. See my **remark** at the end for more details. We get

$$y = 4 - (2x^2 + x + 4) = -2x^2 - x.$$

(3) Stretch in the x-direction by scale factor $\frac{1}{2}$, in terms of transformations, is $f(x) \longrightarrow f(2x)$. So

$$y = -2(2x)^2 - 2x = -8x^2 - 2x.$$

Remark: Reflections in lines of the form $x = a$ and $y = b$ are not covered in much depth at AS Maths, but students have already come across examples of them. Notably, $\sin x = \sin(180 - x)$ comes from symmetry about $x = 90$. In TMUA, these ideas have appeared multiple times already. So this is another topic where students can benefit from proper systematic study for TMUA, and I have a section on this on my site under **additional useful topics**.

Question 15

Find the sum of the solutions of the equation

$$\sqrt{1-\sin^2 x} = 2\sin^2 x - \cos x$$

where $0 \leq x \leq 360°$.

A $180°$

B $360°$

C $540°$

D $720°$

E $900°$

Solution 15

Answer: C

The left-hand side is $\sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$ (note it does not equal to just $\cos x$!).

Using $\sin^2 x = 1 - \cos^2 x$, the right-hand side becomes $2(1-\cos^2 x) - \cos x = 2 - 2\cos^2 x - \cos x$. Let $c = \cos x$, so the equation is $|c| = 2 - 2c^2 - c$.

Case 1: $c \geq 0$ (so $x \in [0°, 90°] \cup [270°, 360°]$). Then $c = 2 - 2c^2 - c \Leftrightarrow c^2 + c - 1 = 0$, giving $c = \frac{-1\pm\sqrt{5}}{2}$.

The minus root is outside of -1 to 1 range of cos. The positive root is a value strictly between 0 and 1, by **considering the graph** of $\cos x$, we deduce it gives two solutions for x, one in 0 to 90, and one in 270 to 360 degree range, precisely matching the case 1 requirements, and the two roots sum to exactly $360°$, again this can be observed from the graph. (Or by basic properties of the cosine function: $\cos x = \cos(x \pm 360n) = \cos(-x)$.) **Case 2**: $c < 0$ (so $x \in (90°, 270°)$). Then $-c = 2 - 2c^2 - c$, i.e. $2c^2 = 2$, so $c = \pm 1$. Only $c = -1$ satisfies $c < 0$, giving $x = 180°$.

Total: 3 solutions. Sum $= 360° + 180° = 540°$.

Remark: On my site, in **additional useful topics**, there is a section on $\sqrt{x^2} = |x|$ to discuss this in more details.

Question 16

Find the maximum value of the function

$$f(x) = \frac{1}{4^x + 4^{-x} - 2\left(2^x + 2^{-x}\right) + 8}.$$

A $\frac{1}{8}$

B $\frac{1}{6}$

C $\frac{1}{5}$

D $\frac{1}{4}$

E 5

F 6

Solution 16

Answer: B

First we need to notice that $4^x + 4^{-x}$ is very close to the square of $2^x + 2^{-x}$, and so this question smells like a completing the square to find max/min question.

On closer inspection: $(2^x + 2^{-x})^2 = 4^x + 2 + 4^{-x}$, therefore, let $u = 2^x + 2^{-x}$, the denominator of the function is transformed to $u^2 - 2u + 6 = (u-1)^2 + 5$. Therefore, at first sight, you may be led to believe that the maximum of $f(x)$ is when the square is 0. However, this occurs when $u = 1$, and **can** u be 1? **No!**

$u = 2^x + 2^{-x}$, when $x = 0$, $u = 2$, as x increases, the 2^x part rapidly tends to infinite, and the 2^{-x} part goes to 0. since the 2^x clearly goes to infinity faster than 2^{-x} go to 0, overall, its not hard to see the sum of the two increases as x increases from 0. Also, u is an even function ($u(x) = u(-x)$), and so it is symmetrical about $x = 0$, therefore we establish the range of u to be from 2 to infinity.

To maxmize f, we want to minimize its denominator, and this now happens at minimum of $(u-1)^2$, which occurs at $u = 2$, which gives the denominator of 6, and the maximum of f is $\frac{1}{6}$.

Remark: an easy way to see the minimum of u is 2, is by sketching and superimposing graphs of 2^x and 2^{-x} on the same plot.

Question 17

Let $0 \leq x \leq 2\pi$. Find the total length of the intervals on which $\sin(7x) \geq \cos(14x)$.

A $\dfrac{\pi}{12}$

B $\dfrac{\pi}{6}$

C $\dfrac{\pi}{3}$

D $\dfrac{2\pi}{3}$

E $\dfrac{5\pi}{12}$

F $\dfrac{\pi}{2}$

G $\dfrac{7\pi}{12}$

H $\dfrac{4\pi}{3}$

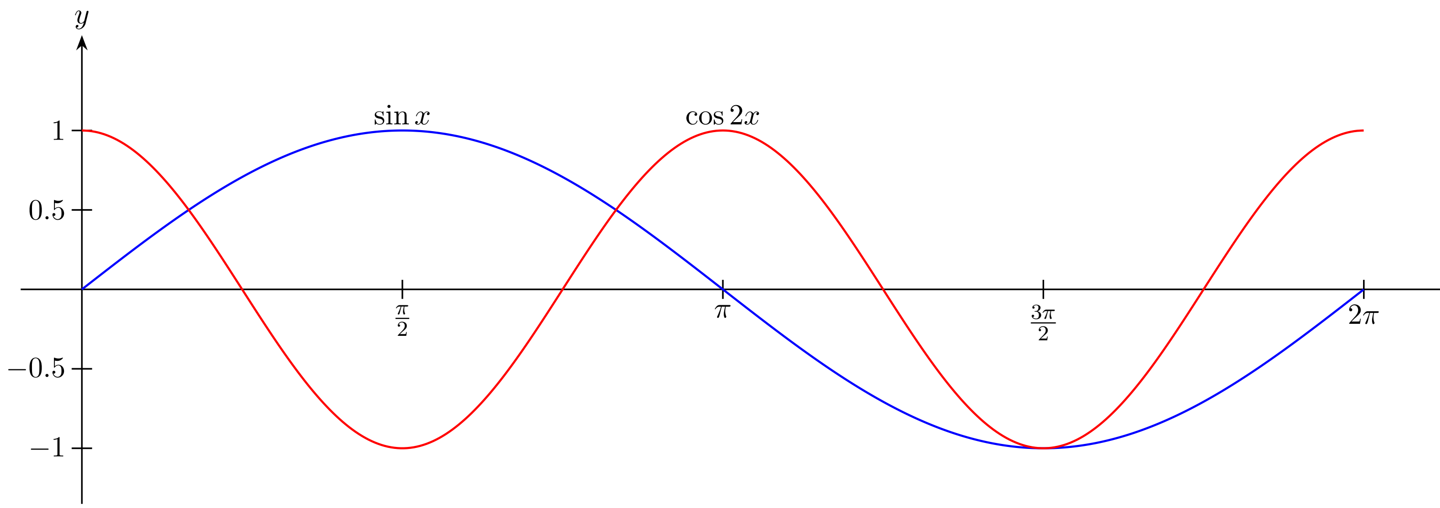
Solution 17

Answer: D

A very important point to observe about this question is that $\sin(7x)$ has 7 **full** periods in the interval 0 to 2π, while $\cos(14x)$ has 14 **full** periods in the interval. Therefore, if we only care about the **length** of the interval for which the inequality is true, it is equivalent to solve $\sin x \geq \cos 2x$.

Hint: If you find it hard to see why the two problems gives the same solution, try sketching graphs of $\sin x$ and $\cos 2x$ on same plot, then graphs of $\sin 2x$ and $\cos 4x$ on another plot.

So it is enough to solve $\sin x \geq \cos 2x$. Easiest way to solve it, is to **sketch** the two graphs on the same plot, you can see that they have intersection at $\pi/6$, which is easily verified too, both give value 0.5 at $\pi/6$. The other intersection can be found by symmetry at $x = 5\pi/6$.



Now that we have the intersection points, looking at the graphs, clearly $\sin x$ is above $\cos 2x$ between these two intersection points, therefore the required interval is

$$\frac{5\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Remark: Initially, I had planned for the question to be something like $\sin(7x - 7\pi/2) \geq \cos(14x - 7\pi)$. This also have the same solution, because horizontal translation does not change the fact that each function still have full periods in 0 to 2π. However, I decided that probably would be too unfair as a question to spot!

Question 18

A function f satisfies

$$\int_0^2 f(x+1)\,dx = 1, \qquad \int_0^2 f(2-x)\,dx = 2, \qquad \int_0^{3/2} f(2x)\,dx = 3.$$

Find the value of $\int_1^2 f(x)\,dx$.

A -6

B -3

C 0

D 1

E 3

F 4

G 5

H 6

Solution 18

Answer: B

Our strategy here is to express each of the 3 integrals as integrals of $f(x)$ over some limits, then deduce its integral between 1 to 2.

For the **first integral**, $f(x+1)$ is a horizontal transformation of $f(x)$. A sneakily quick way to find the corresponding limits is: for $f(x+1)$, when $x=0$, it is $f(1)$, so the integral should start from $f(1)$, and so therefore the equivalent integral of $f(x)$ should start at $x=1$, over same inteval of 2, so end at $x=3$, thus we estabish:

$$\int_0^2 f(x+1)\,dx = \int_1^3 f(x)\,dx = 1.$$

For the **second integral**, I was hesitant to set it in the question, but I felt even though it is unfamiliar to AS maths students, it can be worked out by purely geometrical and transformational arguments, so I set the integral in the question anyway. For this, it is useful to note that because $f(-x)$ is the reflection of $f(x)$ along the y-axis, it follows that in general:

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)dx.$$

It follows that:

$$\int_0^2 f(2-x)\,dx = \int_{-2}^0 f(2+x)\,dx = \int_{-2}^0 f(x+2)\,dx = \int_0^2 f(x)\,dx = 2$$

For the **third integral**, I decided to expose students to another type of horizontal transformation that could, though I think not likely, appear in this type of problem. It can be verified geometrically that the area under $f(x)$ between 0 and $2a$, for example, is exactly twice the area under $f(2x)$ between 0 and a. This leads to the idea that

$$\int_0^{ka} f(x)\,dx = k\int_0^a f(kx)\,dx, \qquad k > 0.$$

This is because $f(kx)$ is a horizontal compression of $f(x)$ by scale factor k, so the widths are divided by k, and therefore the area is divided by k. Applying this to the third integral:

$$\int_0^{3/2} f(2x)\,dx = \frac{1}{2}\times\int_0^3 f(x)\,dx = 3 \quad \Leftrightarrow \quad \int_0^3 f(x)\,dx = 6.$$

Lastly, let $A = \int_0^1 f$, $B = \int_1^2 f$, $C = \int_2^3 f$, and we obtain:

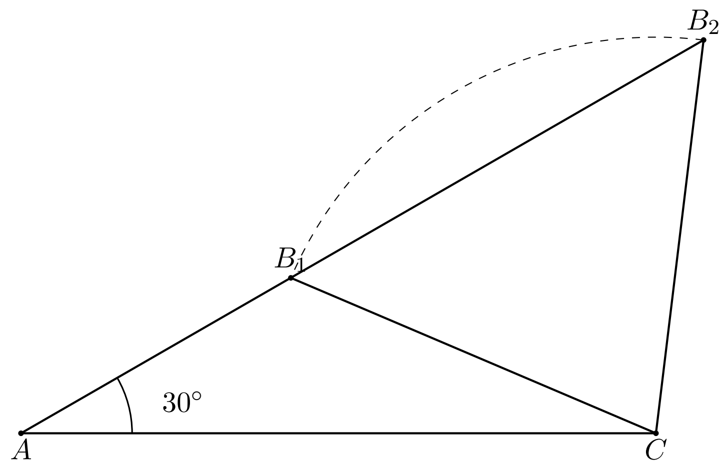
$B + C = 1$, $A + B = 2$ and $A + B + C = 6$.

We need B, and it can be obtained by adding the first two equations subtracting the third, so that $B = 1 + 2 - 6 = -3$.

Remark: these integrals can be easily transformed into integrals of $f(x)$ using integration by substitution. Even though, this is not required knowledge for TMUA, because the question can be done by geometrical considerations, I feel learning the geometrical methods as above is useful for TMUA preparations. On my site, I have a dedicated section for these types of useful extra topics that AS maths students could learn.

Question 19

In a triangle ABC the angle at vertex A is $30°$. The side BC has length $(x-2)(x-3)$ and the side AC has length $(3-x)(x-8)$. Find the complete set of values of x for which there are two non-congruent triangles.



A $3 < x < 4$

B $3 < x < 5$

C $3 < x < 8$

D $4 < x < 5$

E $4 < x < 8$

F $5 < x < 8$

Solution 19

Answer: D

Let $a = BC$ and $b = AC$ so that $a = (x-2)(x-3)$ and $b = (3-x)(x-8) = (x-3)(8-x)$.

Both side lengths must be positive. The condition $a > 0$ gives $x < 2$ or $x > 3$, while $b > 0$ gives $3 < x < 8$. Hence the possible values must first satisfy

$$3 < x < 8.$$

This is the ambiguous SSA case: one angle, the opposite side, and one adjacent side are given. For there to be two non-congruent triangles, the opposite side BC must be longer than the height $AC \sin 30$ but shorter than the adjacent side AC:

$$b \sin 30^\circ < a < b \quad \Leftrightarrow \quad \frac{b}{2} < a < b.$$

Now substitute $a = (x-3)(x-2)$ and $b = (x-3)(8-x)$. Since $3 < x < 8$, we have $x - 3 > 0$.

Now we examine each part separately. First,

$$a < b \quad \Leftrightarrow \quad (x-3)(x-2) < (x-3)(8-x) \quad \Leftrightarrow \quad x - 2 < 8 - x,$$

where in the last step, we divided by $x - 3$ which is positive since $3 < x < 8$. Finally $x - 2 < 8 - x$ means $x < 5$ (i).

Second,

$$\frac{b}{2} < a \quad \Leftrightarrow \quad (x-3)(8-x) < 2(x-3)(x-2) \quad \Leftrightarrow \quad 8 - x < 2x - 4 \quad \Leftrightarrow \quad x > 4 \quad \text{(ii)}.$$

Combining conditions (i) and (ii) with $3 < x < 8$ gives $4 < x < 5$.

Question 20

A sequence of real numbers (a_n) is defined by $a_1 = 2$ and the recurrence

$$a_{n+1}(a_n - 1) = \frac{1}{2}a_n^2 - a_n + \frac{1}{2} \qquad (n \geq 1).$$

Find the value of

$$\sum_{n=1}^{\infty}(a_n + 1).$$

A 0

B 4

C $\frac{5}{2}$

D 6

E 8

F $\frac{9}{4}$

G $\frac{5}{4}$

H ∞

I $-\infty$

Solution 20

Answer: D

It's looking rather complicated...However, here is the first clue, The sum is to infinity! There is a most likely candidate in this case: **convergent geometric sequence**, that is if the answer is finite, $(a_n + 1)$ could be a convergent geometric sequence. Keep this in mind while we simplify the recurrence relation, which obviously is a good starting point.

First, we notice the right-hand side is a square, and factorise it:

$$a_{n+1}(a_n - 1) = \frac{1}{2}a_n^2 - a_n + \frac{1}{2} = \frac{1}{2}(a_n - 1)^2.$$

Then dividing both sides by $a_n - 1$ (we verify $a_n - 1 \neq 0$ later):

$$a_{n+1} = \frac{1}{2}(a_n - 1).$$

Now recall our suspicion that $a_n + 1$ might be a geometric sequence. We can force the issue by making the right-hand side $a_n + 1$ by $+1$ to both sides of the equation:

$$a_{n+1} + 1 = \frac{1}{2}(a_n - 1) + 1 = \frac{1}{2}(a_n + 1).$$

This immediately confirms that $a_n + 1$ is a geometric sequence with $r = \frac{1}{2}$ and first term of 3. Therefore $a_n + 1 = 3 \cdot \left(\frac{1}{2}\right)^{n-1}$. It can also now be easily verified that $a_n - 1 = 0$ has no positive integer solution, so the earlier division is valid.

Finally, the sum to infinity of the sequence $(a_n + 1)$ is given by the usual formula:

$$S_\infty = \frac{a}{1-r} = \frac{3}{1-1/2} = 6$$