JZ Mock Set A Paper 2 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

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Question 1

Consider the equation $x^3 - 3px^2 + 4 = 0$, where p is a real parameter.

Which of the following is a **sufficient** but **not necessary** condition on p for this equation to have exactly one real root?

A $p > 1$

B $p > -1$

C $0 < p < 2$

D $-2 < p < 2$

E $|p| < 1$

F $p^2 < 2$

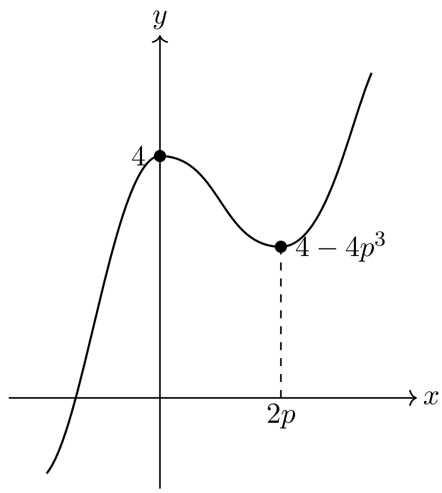
Solution 1

Answer: E

Let $f(x) = x^3 - 3px^2 + 4$. Then $f'(x) = 3x^2 - 6px = 3x(x - 2p)$, with stationary points at $x = 0$ and $x = 2p$, and stationary values

$$f(0) = 4, \qquad f(2p) = 8p^3 - 12p^3 + 4 = 4 - 4p^3.$$

For a cubic with positive leading coefficient, exactly one real root occurs if and only if both stationary values are strictly positive (or both strictly negative). Since $f(0) = 4 > 0$ here, the condition reduces to $f(2p) > 0$, i.e. $4 - 4p^3 > 0 \iff p < 1$. The configuration is shown below.



So $p < 1$ is both **sufficient and necessary**. The interval $|p| < 1$ is a proper subset of $p < 1$, so it is a **sufficient but not necessary** condition. Indeed, $-1 < p < 1$ guarantees $p < 1$, hence it is sufficient, but not a necessity, since for example $p = -2$ also gives exactly one real root. Confirming it is **sufficient but not necessary**.

Lastly, though not needed, we note the other options are not proper subsets of $p < 1$, therefore none of them are sufficient, as they contain values of p that does not lead to one real root.

Question 2

What is the gradient of the curve

$$y = \frac{(\sqrt{x}+2)^3}{x\sqrt{x}}$$

at the point where $x = 4$?

A $-\dfrac{5}{2}$

B $-\dfrac{3}{2}$

C $-\dfrac{7}{2}$

D $\dfrac{7}{2}$

E $\dfrac{5}{2}$

F $\dfrac{3}{2}$

Solution 2

Answer: B

Expand the numerator using the binomial expansion:

$$(\sqrt{x}+2)^3 = x^{3/2} + 6x + 12\sqrt{x} + 8.$$

Divide each term by $x\sqrt{x} = x^{3/2}$ to put y in power form:

$$y = 1 + 6x^{-1/2} + 12x^{-1} + 8x^{-3/2}.$$

Differentiate term by term:

$$\frac{dy}{dx} = -3x^{-3/2} - 12x^{-2} - 12x^{-5/2}.$$

At $x = 4$ we have $4^{-3/2} = \frac{1}{8}$, $4^{-2} = \frac{1}{16}$, $4^{-5/2} = \frac{1}{32}$, so

$$\frac{dy}{dx} = -\frac{3}{8} - \frac{3}{4} - \frac{3}{8} = -\frac{3}{2}.$$

Question 3

A student attempts to solve the inequality

$$\sqrt{x-3} < x-2.$$

Consider the following attempt:

$$\text{Can only square-root non-negative, therefore } x-3 \geq 0 \quad \Leftrightarrow \quad x \geq 3. \qquad \text{(I)}$$

$$\text{Squaring both sides gives } x-3 < (x-2)^2. \qquad \text{(II)}$$

$$\text{Simplifies to } 0 < x^2-5x+7. \qquad \text{(III)}$$

$$\text{The discriminant of } x^2-5x+7 \text{ is } 25-28=-3<0. \qquad \text{(IV)}$$

$$\text{Since the leading coefficient is positive, } x^2-5x+7>0 \text{ for all real } x \geq 3. \qquad \text{(V)}$$

$$\text{Therefore the solution is } x \geq 3. \qquad \text{(VI)}$$

A The attempt is invalid, and the first problematic step is (I).

B The attempt is invalid, and the first problematic step is (II).

C The attempt is invalid, and the first problematic step is (III).

D The attempt is invalid, and the first problematic step is (IV).

E The attempt is invalid, and the first problematic step is (V).

F The attempt is invalid, and the first problematic step is (VI).

G The attempt is valid, and the solution is correct.

Solution 3

Answer: G

The attempt is valid, and the solution is correct!

The **trap** in this question is that step **(II)** may look careless. Students are often under the impression that we cannot square both sides of an inequality, so they may wrongly think that step **(II)** is automatically invalid.

However, the true rule is more nuanced. Whether squaring is valid depends on whether the original inequality and the squared inequality are **equivalent**.

In general, they are not equivalent. But in this case, step **(I)** gives $x \geq 3$, so $x - 2 \geq 1 > 0$. Also, $\sqrt{x-3}$ is non-negative. Therefore both sides of the inequality are non-negative, and squaring both sides gives an equivalent inequality:

$$\sqrt{x-3} < x - 2 \Leftrightarrow x - 3 < (x-2)^2.$$

So step **(II)** is a valid and useful step, not an error. The rest of the working correctly shows that $x^2 - 5x + 7 > 0$ for all real x, so the inequality holds for every x in the domain $x \geq 3$.

Question 4

Consider the following statement:

If a positive integer N has the property that N^2 is divisible by 12, then N is divisible by 12.

Which of the following are counterexamples to this statement?

I $N = 6$

II $N = 8$

III $N = 18$

IV $N = 24$

A none of them

B I only

C II only

D I and III only

E I, II and III only

F II and IV only

G I, III and IV only

H I, II, III and IV

Solution 4

Answer: D

A counterexample to "if P then Q" is a value for which P holds but Q fails. Here P is "N^2 is divisible by 12" and Q is "N is divisible by 12".

Another useful way to think about this is that the statement is only making a claim about those N with property N^2 is divisible by 12.

I: $N = 6$. $N^2 = 36 = 12 \cdot 3$, so P holds. $12 \nmid 6$, so Q fails. **Counterexample.**

II: $N = 8$. $N^2 = 64$, and $64 = 12 \cdot 5 + 4$, so $12 \nmid 64$. The P is false, therefore this N **does not** qualify, the statement does not apply to it, and it is not a counterexample.

III: $N = 18$. $N^2 = 324 = 12 \cdot 27$, so P holds. $12 \nmid 18$, so Q fails. **Counterexample.**

IV: $N = 24$. $N^2 = 576 = 12 \cdot 48$, so P holds. But $24 = 12 \cdot 2$, so Q also holds. The implication is satisfied here, so this is **not** a counterexample.

Only I and III are counterexamples.

Question 5

A student attempts to prove the following **claim:**

For every positive integer n, if $n^2 + 2$ is prime, then n is a multiple of 3.

The student's argument is given line by line below.

I. Suppose n is a positive integer such that $n^2 + 2$ is prime, but n is not a multiple of 3.

II. Then the remainder of n on division by 3 is 1 or 2.

III. In either case, n^2 leaves remainder 1 when divided by 3.

IV. Therefore $n^2 + 2$ leaves remainder 0 when divided by 3.

V. So 3 divides $n^2 + 2$.

VI. Hence $n^2 + 2 = 3k$ for some integer k.

VII. It follows that $n^2 + 2$ is composite and not prime.

VIII. This contradicts the assumption in (I) that $n^2 + 2$ is prime.

IX. Therefore n is a multiple of 3.

Which one of the following statements about the argument is correct?

A The argument is completely correct.

B The first error in the argument is on line I.

C The first error in the argument is on line II.

D The first error in the argument is on line III.

E The first error in the argument is on line IV.

F The first error in the argument is on line V.

G The first error in the argument is on line VI.

H The first error in the argument is on line VII.

I The first error in the argument is on line VIII.

J The first error in the argument is on line IX.

Solution 5

Answer: H

Lines I to V are correct. Line I sets up the contradiction. Line II uses the standard fact that every integer leaves remainder 0, 1 or 2 on division by 3, and (I) rules out 0. Line III: if $n = 3m + 1$ then $n^2 = 9m^2 + 6m + 1$, remainder 1; if $n = 3m + 2$ then $n^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$, remainder 1 again. Line IV adds 2 to a remainder of 1 to get 3, i.e. remainder 0. Line V is just (IV) restated in divisibility language.

Line VII is where the error occurs, $3k$ is **not always** composite, $k = 1$ and you get $3k = 3$ which is prime! Therefore the first error is on VII.

Remark 1: It turns out the claim is in fact true for all positive integers n except for $n = 1$, and the proof is also valid for all but $n = 1$. When $n = 1$, $n^2 + 2 = 3$ is indeed a prime, but $n = 1$ is not a multiple of 3.

Remark 2: In this question, some basic knowledge of **modulo arithmetic** is not required, but very helpful - get to the conclusions faster. This is a topic discussed in the **additional useful topics** section of my site.

Question 6

Of the 200 books in a school library: 140 are hardback and 60 are paperback. 70 are illustrated and 130 are not illustrated. 170 are written in English and 30 are not written in English.

What is the smallest number of books that could be hardback, not illustrated, and written in English?

A 10

B 20

C 30

D 40

E 50

F 60

G 70

Solution 6

Answer: D

Let H be the set of hardback books, N be the set of not illustrated books, and E be the set of books written in English.

We want the smallest possible value of

$$|H \cap N \cap E|.$$

Instead, consider the books that do not belong to this group. A book is not in $H \cap N \cap E$ if it is paperback, illustrated, or not written in English.

There are 60 paperback books, 70 illustrated books, and 30 books not written in English. Therefore, the maximum possible number of books outside the group is

$$60 + 70 + 30 = 160.$$

Note that this maximum is only attainable, if these three groups of books have no overlaps, for example, there must be no book which is both paperback and illustrated.

So the minimum possible number of books inside the group is

$$200 - 160 = 40.$$

Once more, this is possible, for example, if the paperback books, illustrated books, and non-English books are all separate groups.

Therefore the smallest possible number of books that are hardback, not illustrated, and written in English is 40.

Alternative set algebra method:

We want the smallest possible value of

$$|H \cap N \cap E|.$$

The complement is

$$(H \cap N \cap E)^c = H^c \cup N^c \cup E^c.$$

Now

$$|H^c| = 60, \qquad |N^c| = 70, \qquad |E^c| = 30.$$

So by the union bound,

$$|H^c \cup N^c \cup E^c| \leq 60 + 70 + 30 = 160.$$

Since $|\Omega| = 200$,

$$200 - |H \cap N \cap E| \leq 160.$$

Therefore

$$|H \cap N \cap E| \geq 40.$$

So the smallest possible number for $|H \cap N \cap E|$ is 40.

Question 7

The non-zero real numbers a, b, c and d satisfy $a < b$ and $c < d$.

Which of the following statements are **necessarily** true?

I $\dfrac{1}{a^3} > \dfrac{1}{b^3}$.

II $3^{-a} < 3^{-b}$.

III If a, b are negative, then $ac < bd$.

IV If a, b, c, d are all negative, then $ac > bd$.

A none of them

B **II** only

C **III** only

D **II** and **IV** only

E **I**, **II** and **III** only

F **II**, **III** and **IV** only

G **I**, **II** and **IV** only

H **IV** only

I **I** only

J **III** and **IV** only

Solution 7

Answer: H

For **I**, this is necessarily true **if** a and b are both positive, but without the additional constraint, not true in general. A counterexample is $a = -1$ and $b = 1$.

For **II**, since $a < b$, multiplying by -1 gives $-a > -b$. Since 3^x is an increasing function, $3^{-a} > 3^{-b}$ must be true, and the statement stated with the reverse sign is never true.

For **III**, this is necessarily true **if** in addition, b and d are positive, but without the additional constraint, not true in general. A counterexample is $a = -2$, $b = -1$, $c = -4$ and $d = -3$.

For **IV**, this is necessarily true. We have $a < b < 0$ and $c < d < 0$. Since $c < 0$, multiplying $a < b$ by c reverses the inequality, so

$$ac > bc.$$

Since $b < 0$, multiplying $c < d$ by b also reverses the inequality, so

$$bc > bd.$$

Therefore $ac > bc > bd$ and hence $ac > bd$.

Only **IV** is necessarily true.

Question 8

A sequence is defined by $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{1 + a_n}$$

for $n \geq 1$. Find the value of

$$\sum_{n=1}^{\infty} a_n a_{n+1}.$$

A 0

B 1

C 2

D 4

E $\frac{5}{2}$

F $\frac{3}{2}$

G ∞

Solution 8

Answer: B

Take reciprocals of the recurrence: from $a_{n+1} = \dfrac{a_n}{1 + a_n}$ we get

$$\frac{1}{a_{n+1}} = \frac{1 + a_n}{a_n} = \frac{1}{a_n} + 1.$$

Think of $\dfrac{1}{a_n}$ as b_n, then $b_{n+1} = b_n + 1$ is an arithmetic sequence, and $b_1 = 1$. Therefore $b_n = n$, and $a_n = \dfrac{1}{n}$.

Remark 1: You could also manually find the first a few terms of a_n to guess that $a_n = \dfrac{1}{n}$.

Now $a_n a_{n+1} = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, so that when you write out:

$$\sum_{n=1}^{\infty} a_n a_{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots$$

We see that the terms in the tail go towards 0, and cancels pairwise, therefore the answer is 1.

Remark 2: using only $a_n a_{n+1} = \dfrac{1}{n(n+1)}$, if you compute the sum of the first a few terms, in sequence, you will find that the partial sums are: 3/4, 4/5, 5/6..., which also leads to conclusion the full sum is 1.

Question 9

Find the complete set of values of k for which the graph $y = |2x + k|$ lies strictly below the curve $y = x^2 + (k-1)x + 5$ for every real value of x.

A $1 - 2\sqrt{3} < k < 1 + 2\sqrt{3}$

B $k < 1 + 2\sqrt{7}$

C $1 - 2\sqrt{5} < k < 1 + 2\sqrt{5}$

D $1 - 2\sqrt{3} < k < 1 + 2\sqrt{5}$

E $k < 1 - 2\sqrt{3}$ or $k > 1 + 2\sqrt{3}$

F There are no such values of k.

G $k < 1 - 2\sqrt{5}$ or $k > 1 + 2\sqrt{5}$

Solution 9

Answer: A

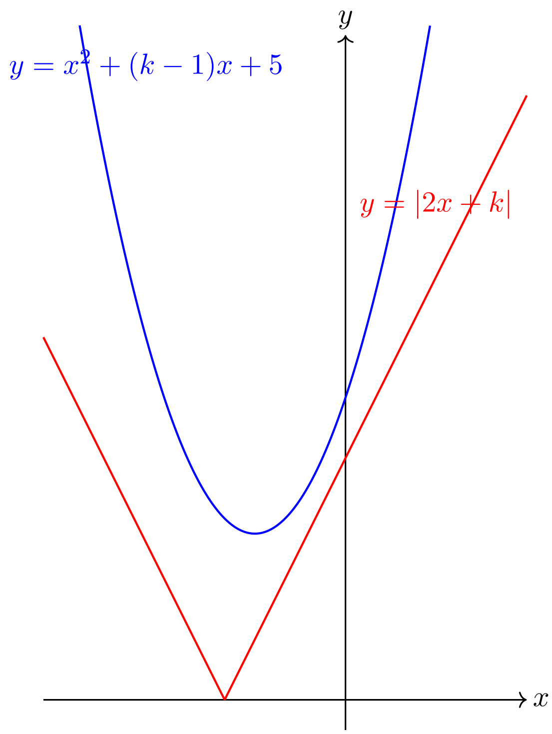
The line $y = |2x + k|$ lies strictly below the curve if and only if:

$$|2x + k| < x^2 + (k-1)x + 5 \quad \text{for all } x \in \mathbb{R},$$

which is equivalent to:

$$2x + k < x^2 + (k-1)x + 5 \text{ and } -(2x+k) < x^2 + (k-1)x + 5 \quad \text{for all } x \in \mathbb{R}.$$

This is because $|A| < B$ if and only if $A < B$ and $-A < B$. Geometrically, the parabola must dominate both branches of the V.



Rearranging gives

$$x^2 + (k-3)x + (5-k) > 0 \quad \text{and} \quad x^2 + (k+1)x + (5+k) > 0 \quad \text{for all } x \in \mathbb{R}.$$

Each has positive leading coefficient, so each holds everywhere if and only if its discriminant is strictly negative.

First discriminant:

$$(k-3)^2 - 4(5-k) = k^2 - 2k - 11 < 0 \Leftrightarrow 1 - 2\sqrt{3} < k < 1 + 2\sqrt{3}.$$

Second discriminant:

$$(k+1)^2 - 4(5+k) = k^2 - 2k - 19 < 0 \Leftrightarrow 1 - 2\sqrt{5} < k < 1 + 2\sqrt{5}.$$

We require both to be true, therefore:

$$1 - 2\sqrt{3} < k < 1 + 2\sqrt{3}.$$

Question 10

The region R in the (x, y)-plane consists of all points satisfying **both**

$$|y - x^2| < 2 \qquad \text{and} \qquad x + y < 4.$$

Which of the following claims about points in R are true?

I: For every $(x, y) \in R$, $y \leq 6$.

II: For every $(x, y) \in R$, $y > -2$.

III: For every $(x, y) \in R$, $x \leq 2$.

A None of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Solution 10

Answer: G

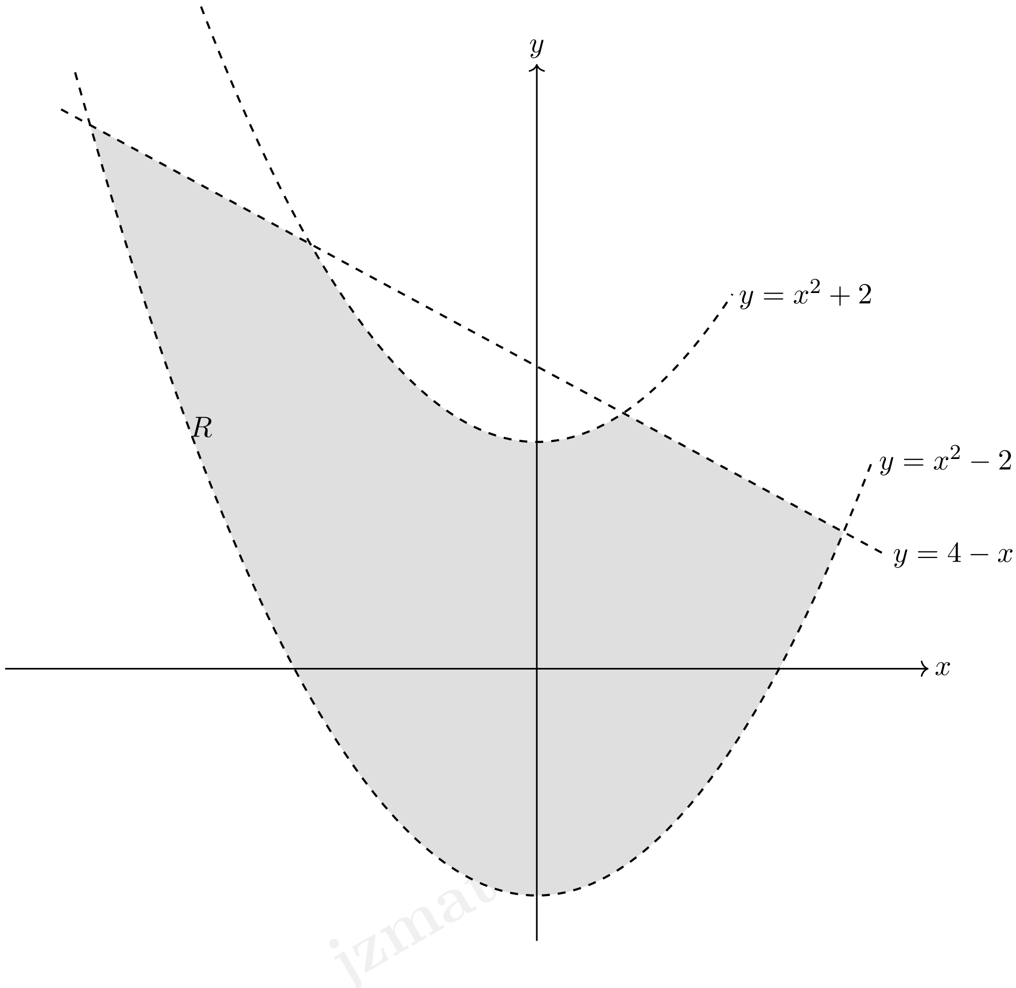
Rewrite the defining conditions of R as

$$x^2 - 2 < y < x^2 + 2 \qquad \text{and} \qquad y < 4 - x.$$

Rewrite further as

$$x^2 - 2 < y \qquad \text{and} \qquad y < x^2 + 2 \qquad \text{and} \qquad y < 4 - x.$$

Now make a sketch of the region R:



It is now immediate that **II** is true, since the minimum of $x^2 - 2$ is -2. For **I** and **III**, we clearly need to find the intersection points of $x^2 - 2$ and $4 - x$.

$$x^2 - 2 = 4 - x \quad \Leftrightarrow \quad x = 2, -3.$$

Therefore **III** is true.

Next when $x = -3$, $y = 7$, this point itself is not in the region, but it is on the boundary, so there are clearly points in R with y-coordinate between 6 and 7, therefore **I** is false.

Question 11

P: k is an integer multiple of π.

Q: $\int_0^k \left(\cos x + \cos^3(2x)\right)\, dx = 0$.

Which of the following best describes the logical relationship between P and Q?

A P is necessary and sufficient for Q.

B P is necessary but not sufficient for Q.

C P is sufficient but not necessary for Q.

D P is neither necessary nor sufficient for Q.

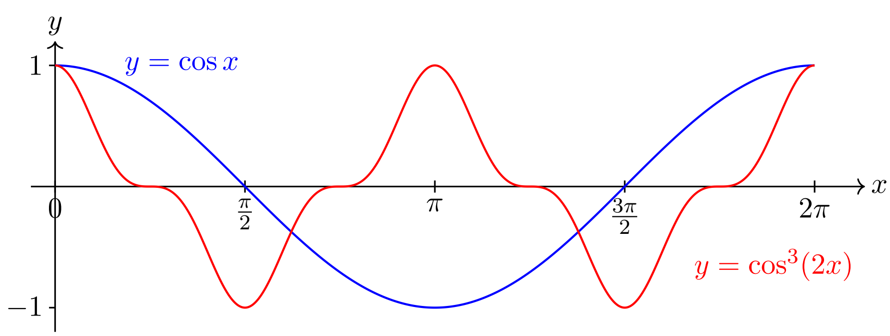
Solution 11

Answer: A

Consider the graph of $\cos x$, we can deduce from the graph that $\int_0^k \cos x dx = 0$ if and only if k is a multiple of π.

Consider the graph of $\cos 2x$, we can deduce from the graph that $\int_0^k \cos(2x)dx = 0$ if and only if k is a multiple of $\frac{\pi}{2}$, and therefore $\int_0^k \cos^3(2x)dx = 0$ also if and only if k is a multiple of $\frac{\pi}{2}$.

Together, it is clear that P is sufficient for Q.



Now, let's consider if it is possible for Q to be true while k is not a multiple of π. The answer is no: Because we can observe from the graphs that $\int_0^\pi \left(\cos x + \cos^3(2x)\right)\, dx = \int_\pi^{2\pi} \left(\cos x + \cos^3(2x)\right)\, dx = 0$, and likewise we can see that every integral of inteval length π, starting at a multiple of π is 0. This, together with observing the graph, it becomes clear that P is also necessary for Q.

Therefore P and Q are equivalent, and they are necessary and sufficient for each other.

Remark: Here is a more rigorous argument of the last part if you are interested.

Let $k = n\pi + r$, where n is an integer and $0 \leq r < \pi$.

If $r = 0$, then k is an integer multiple of π, so P is true.

If $0 < r < \pi$, then from the graph, the accumulated signed area over this incomplete interval is not zero. It only returns to zero at the next multiple of π. Therefore Q cannot be true when $0 < r < \pi$.

Hence Q is true only when k is an integer multiple of π. So Q implies P.

Question 12

Consider the curve

$$y = \log_{x-1} 4 \quad \text{for } x > 1,\ x \neq 2.$$

Which of the following statements about this curve are true?

I $\quad y > 0$ for every x in the domain.

II $\quad y$ is a strictly decreasing function of x on the domain.

III $\quad$ For every real number y, there exists a value of x in the domain such that $y = \log_{x-1} 4$.

IV $\quad y = 0$ is an asymptote.

A $\quad$ I only

B $\quad$ II only

C $\quad$ III only

D $\quad$ IV only

E $\quad$ I and II only

F $\quad$ I and IV only

G $\quad$ II and III only

H $\quad$ III and IV only

I $\quad$ None of the statements is true.

J $\quad$ All four statements are true.

Solution 12

Answer: D

Rewrite $y = \log_{x-1} 4$ as $(x-1)^y = 4$. Taking natural logarithms gives

$$y \ln(x-1) = \ln 4.$$

Since $x \neq 2$, we have $\ln(x-1) \neq 0$, so

$$y = \frac{\ln 4}{\ln(x-1)}.$$

I. Take $x = \frac{3}{2}$. Then $\ln(x-1) = \ln\frac{1}{2} < 0$, so $y < 0$. Therefore, statement I is false.

II. The function is strictly decreasing on each of the intervals $(1,2)$ and $(2,\infty)$, since the denominator is increasing and does not change sign within either interval. However, $x = 2$ is a vertical asymptote, because $\ln(x-1) \to 0$ as $x \to 2$. Moreover, the function has opposite signs on the two sides of the asymptote. For example, $x = \frac{3}{2}$ gives $y = -2$, while $x = 3$ gives $y = 2$. Since $\frac{3}{2} < 3$ but $-2 < 2$, the function is not strictly decreasing on its entire domain. Therefore, statement II is false.

III. The numerator $\ln 4$ is non-zero, so

$$\frac{\ln 4}{\ln(x-1)}$$

can never equal 0. Therefore, there is no value of x for which $y = 0$, so statement III is false.

IV. As $x \to \infty$, we have $\ln(x-1) \to \infty$, and hence

$$\frac{\ln 4}{\ln(x-1)} \to 0.$$

Therefore, $y = 0$ is a horizontal asymptote, so statement IV is true.

Hence only statement IV is true.

Question 13

Find the maximum value of
$$f(x) = \frac{2\sin^2 x + 10\cos x - 14}{\cos^2 x + 3\cos x - 10}$$
where x is a real number.

A $\frac{2}{3}$

B 1

C $\frac{5}{3}$

D 2

E $\frac{8}{3}$

F f has no maximum value

Solution 13

Answer: D

Let $u = \cos x$, so $u \in [-1, 1]$ and $\sin^2 x = 1 - u^2$.

The numerator becomes

$$2(1-u^2) + 10u - 14 = -2u^2 + 10u - 12 = -2(u-2)(u-3).$$

The denominator becomes
$$u^2 + 3u - 10 = (u-2)(u+5).$$

Since $u \in [-1, 1]$ we have $u \neq 2$, so the factor $(u-2)$ cancels:

$$f = \frac{-2(u-3)}{u+5} = \frac{6-2u}{u+5} = -2 + \frac{16}{u+5}.$$

Therefore maximum is attained at $u = -1$, and it is $-2 + 16/4 = 2$.

Question 14

Given curve $y = -\frac{1}{3}x^3 + a^2x + a$, has a local minimum in the second quadrant, find the possible values of a.

A $0 < a < \sqrt{\frac{3}{2}}$

B Any a such that $a \neq 0$

C $-\sqrt{\frac{3}{2}} < a < \sqrt{\frac{3}{2}}$ with $a \neq 0$

D $-\sqrt{3} < a < 0$

E $0 < a < \sqrt{3}$

F $-\sqrt{3} < a < \sqrt{3}$ with $a \neq 0$

G There are no possible values of a.

Solution 14

Answer: A

First note $a \neq 0$ or else $y = 1/3x^3$ has no local minimum.

Differentiate: $y' = -x^2 + a^2 = -(x-a)(x+a)$, so the stationary points are at $x = a, -a$ with corresponding $y = \frac{2}{3}a^3 + a, -\frac{2}{3}a^3 + a$.

Due to the shape of the cubic, the local minimum must be the one with the smaller x-coordinate.

Suppose $a < 0$, then the 2nd quadrant minimum has $x = a$, but its $y = \frac{2}{3}a^3 + a < 0$, which puts it in 3rd quadrant, so this can't work.

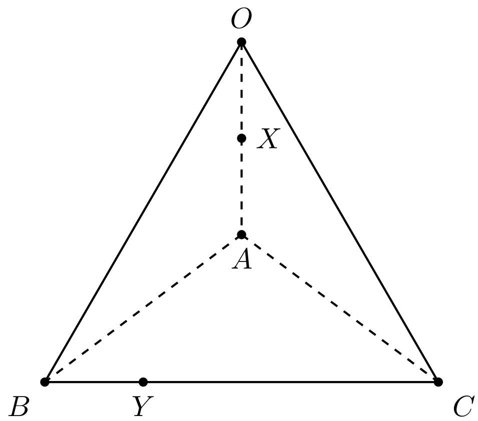
Therefore $a > 0$, and the 2nd quadrant minimum has $x = -a$, and we need its $y > 0$:

$$-\frac{2}{3}a^3 + a > 0 \quad \Leftrightarrow \quad -\frac{2}{3}a^2 + 1 > 0 \quad \Leftrightarrow \quad 0 < a < \sqrt{\frac{3}{2}}.$$

In the last steps, we are relying on $a > 0$.

Question 15

A regular tetrahedron $OABC$ has every edge of length 12 metres. The point X is the midpoint of edge OA, and the point Y lies on edge BC with $BY = 3$ metres, as shown below.



What is the shortest distance, in metres, from X to Y measured entirely along the outer surface of the tetrahedron?

A $3\sqrt{3}$

B $3\sqrt{7}$

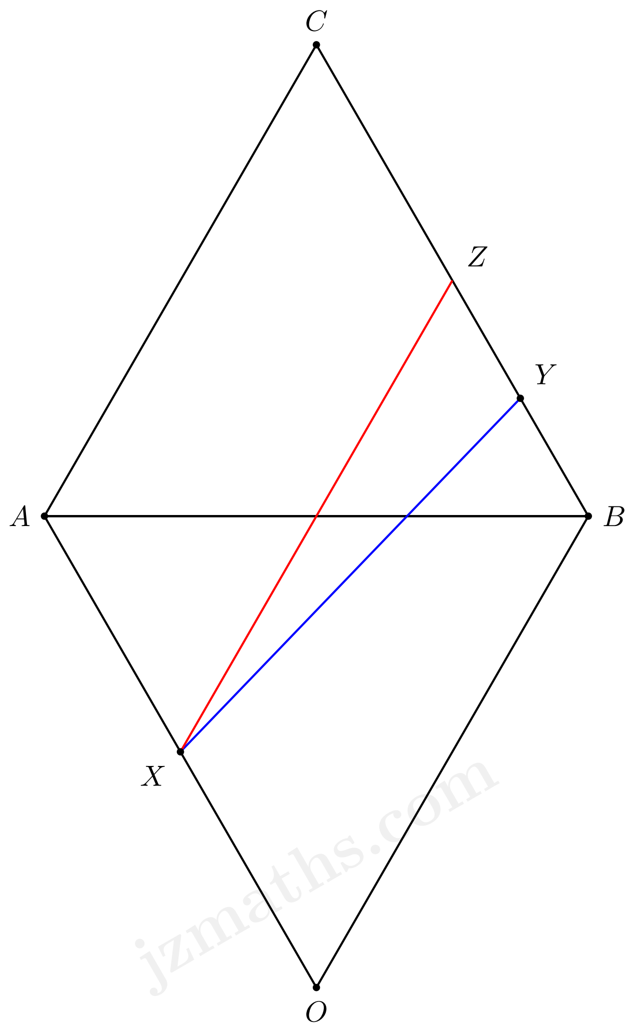
C $3\sqrt{13}$

D $3\sqrt{21}$

E 21

Solution 15

Answer: C



Unfold and flatten the two faces ABC and ABO along their common edge AB, as shown above. The shortest path across these two faces is then the straight line segment XY.

Let Z be the midpoint of BC. Since X and Z are the midpoints of OA and BC respectively, XZ is parallel and equal in length to AC, so $XZ = 12$. Also, $BZ = 6$ and $BY = 3$, so $ZY = 3$.

Since XZ is parallel to AC, we have $\angle XZY = 60^\circ$. Applying the cosine rule to triangle XZY gives

$$XY^2 = 12^2 + 3^2 - 2(12)(3)\cos 60^\circ = 117.$$

Therefore $XY = \sqrt{117} = 3\sqrt{13}$.

Question 16

A safe has three levers, A, B and C, each of which can be positioned either left or right at any particular time. The state of the safe (open or closed) depends only on the positions of these three levers. It is known that:

If lever A is right and (lever B is left or lever C is right), **then** the safe is open.

Which one of the following statements **must** be true?

A If the safe is open, then lever A is right and either lever B is left or lever C is right.

B If the safe is closed, then lever A is left, and either lever B is right or lever C is left.

C If the safe is closed, then lever A is left, lever B is right and lever C is left.

D If the safe is closed, then either lever A is left, or both lever B is right and lever C is left.

E If lever A is left, or both lever B is right and lever C is left, then the safe is closed.

F If the safe is open, then either lever A is left, or both lever B is right and lever C is left.

Solution 16

Answer: D

Let P be the statement: lever A is right and (lever B is left or lever C is right).

Let Q be the statement: the safe is open.

Then the **only** truth we are given is P implies Q. In particular, we do not know if P is true or false, nor do we know if Q is true or false.

A: Q implies P, which need not be true.

It is worth noting there are two statements equivalent to P imples Q which are likely to be our candidate correct statements:

"not Q implies not P", the contrapositive. OR "P and not Q cannot both both true".

B: not Q implies something which is not the negation of P, so it is not the contrapositive, and not true.

Note that the negation of P is actually: "either lever A is left, or both lever B is right and lever C is left".

C: not Q implies something which is not the negation of P, so it is also not true.

D: not Q implies not P, this is the contrapositive, it is equivalent to P implies Q, so it must be true. **This is our answer**, but for completeness let's examing the rest.

E: not P implies not Q, which does not follow from P implies Q, so it is not true.

F: Given Q, we are supposed to be able to deduce something, but nothing can be deduced given Q is true. So this statement is also false.

Remark: The trick that makes this question easy, is to reduce it to its pure logical form of P implies Q, and understanding that from this, we can only deduce two equivalent statements:

(1) the **contrapositive**: not Q implies not P.

(2) what is sometimes called the **negation form**: P and not Q cannot be both true.

Question 17

In this question, n is an integer greater than 1.

Statement: If n does not divide $k!$ for any positive integer k with $k < n$, then n is prime.

The following is an attempted proof of the statement:

(1) Assume n does not divide $k!$ for any positive integer k with $k < n$.

(2) Suppose for contradiction that n is composite, so $n = ab$ for some integers a, b with $1 < a \leq b < n$.

(3) Case 1: $a < b$. Then a and b are distinct integers, each in the set $\{2, 3, \ldots, n-1\}$.

(4) Therefore a and b both appear as factors in the product $(n-1)! = 1 \cdot 2 \cdots (n-1)$, so $ab = n$ divides $(n-1)!$.

(5) Case 2: $a = b$, so $n = a^2$.

(6) Since $a \geq 2$, the integers a and $2a$ both satisfy $a \geq 2$ and $2a \geq 4$, so they are positive integers.

(7) Since $a \geq 2$, we have $a < a^2 = n$ and $2a < a^2 = n$, so a and $2a$ are distinct integers in the set $\{2, 3, \ldots, n-1\}$.

(8) Therefore a and $2a$ both appear as factors in $(n-1)!$, so $a \cdot 2a = 2a^2 = 2n$ divides $(n-1)!$, and in particular n divides $(n-1)!$.

(9) In both cases n divides $(n-1)!$, contradicting the assumption in line 1. Hence n is prime.

A First mistake appears in line 1 and the statement is false.

B First mistake appears in line 2 and the statement is false.

C First mistake appears in line 3 and the statement is false.

D First mistake appears in line 4 and the statement is false.

E First mistake appears in line 5 and the statement is false.

F First mistake appears in line 6 and the statement is false.

G First mistake appears in line 7 and the statement is false.

H The proof is valid and the statement is true.

Solution 17

Answer: G

Line 7 asserts that $a \geq 2$ implies

$$2a < a^2 = n.$$

This is wrong. The inequality $2a < a^2$ is equivalent to $a > 2$, not merely $a \geq 2$.

A counterexample to the statement is $n = 4$. We have

$$1! = 1, \qquad 2! = 2, \qquad 3! = 6,$$

and none of these is divisible by 4, yet 4 is not prime. So the statement is false.

Tracing $n = 4$ through the proof, we have $a = b = 2$, so this falls into Case 2. Lines 1 to 6 are all valid, but line 7 claims that $2a < n$. Here this would say $4 < 4$, which is false.

Therefore the first mistake appears in line 7, and the statement is false.

Question 18

Let $a, b, c > 0$ with $a \neq 1$, $b \neq 1$ and $c \neq 1$. Consider the three equations

$$\log_a b = c, \qquad \log_b c = a, \qquad \log_c a = b.$$

Which one of the following statements about the solutions (a, b, c) of this system is correct?

A The equations specify a, b and c uniquely.

B The equations specify the product abc uniquely but have infinitely many solutions for (a, b, c).

C The equations specify exactly one of a, b, c uniquely but have infinitely many solutions for the other two.

D The equations have no solutions.

Solution 18

Answer: D

We are given

$$\log_a b = c, \qquad \log_b c = a, \qquad \log_c a = b.$$

Since $a, b, c > 0$, the right hand sides are all positive. Therefore

$$\log_a b > 0, \qquad \log_b c > 0, \qquad \log_c a > 0.$$

This means that in each logarithm, the base and the argument must be on the **same side of 1**.

So a and b are on the same side of 1, b and c are on the same side of 1, and c and a are on the same side of 1.

Therefore either $a, b, c > 1$ or $0 < a, b, c < 1$.

Now rewrite the equations in exponential form: $b = a^c, \qquad c = b^a, \qquad a = c^b.$

First suppose

$$a, b, c > 1.$$

Then $c > 1$, so $b = a^c > a$. Similarly, in a cyclic way, $c > b$ and $a > c$. Therefore $a < b < c < a$, this is not possible.

Therefore it must be the case that

$$0 < a, b, c < 1.$$

For a number between 0 and 1, raising it to a power less than 1 makes it larger. Since $c < 1$, $b = a^c > a$, and similarly $c > b$ and $a > c$. So we get $a < b < c < a$, which again is impossible.

Therefore there are no solutions.

Question 19

Let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x. Compute

$$\int_0^{20} \lceil x \rceil \cdot 2^{\lceil x \rceil}\, dx.$$

A $20 \cdot 2^{21}$

B $19 \cdot 2^{21}$

C $18 \cdot 2^{20} + 2$

D $19 \cdot 2^{21} + 2$

E $19 \cdot 2^{20} + 2$

F $20 \cdot 2^{20} + 2$

Solution 19

Answer: D

On the interval $(k-1, k]$ we have $\lceil x \rceil = k$, so the integrand equals $k \cdot 2^k$ on each unit interval. Hence

$$\int_0^{20} \lceil x \rceil \cdot 2^{\lceil x \rceil}\, dx = \sum_{k=1}^{20} k \cdot 2^k.$$

To evaluate $S = \sum_{k=1}^{20} k \cdot 2^k$, use the standard trick of subtracting a shifted copy:

$$2S = \sum_{k=1}^{20} k \cdot 2^{k+1} = \sum_{j=2}^{21} (j-1) \cdot 2^j.$$

Then

$$S = 2S - S = -\sum_{j=2}^{20} 2^j + 20 \cdot 2^{21} - 1 \cdot 2^1.$$

The geometric sum is $\sum_{j=2}^{20} 2^j = 2^{21} - 4$, so

$$S = -(2^{21} - 4) + 20 \cdot 2^{21} - 2 = 19 \cdot 2^{21} + 2.$$

Question 20

Given that p is an integer, find the minimum value of p such that the equation

$$\sqrt{\frac{1+\sin x}{1-\sin 2x}} = \sqrt{\frac{1+\sin 2x}{1-\sin x}}$$

has exactly 16 solutions in the interval $0° \leq x < p°$.

A 540

B 541

C 720

D 721

E 900

F 901

G 1080

H 1081

Solution 20

Answer: F

The equation is equivalent to

$$\sqrt{1-\sin^2 x} = \sqrt{1-\sin^2 2x},$$

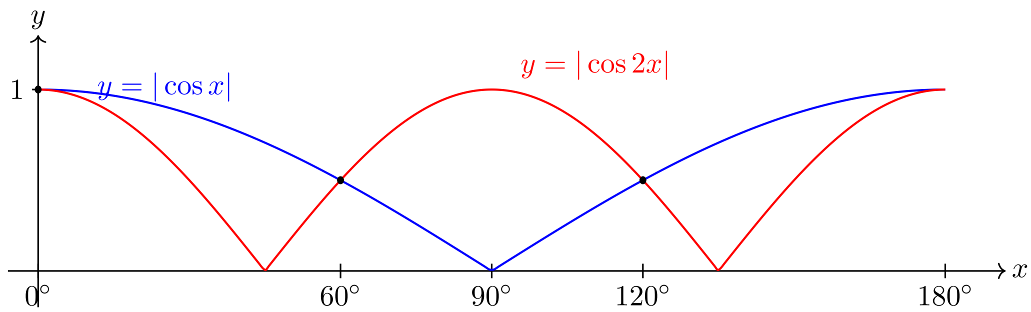
or

$$\sqrt{\cos^2 x} = \sqrt{\cos^2 2x},$$

which is the same as

$$|\cos x| = |\cos 2x|.$$

Yes yes, this is another question that I am reminding students the same old $\sqrt{x^2} \neq x$, but equals to $|x|$!



Now consider the graphs of $y = |\cos x|$ and $y = |\cos 2x|$. Both graphs repeat every $180°$, so it is enough to count the intersections in one interval of length $180°$.

On $0° \le x < 180°$, the graph of $y = |\cos x|$ makes one V-shape, while $y = |\cos 2x|$ makes two V-shapes. From the graph, they intersect 3 times in each interval of the form

$$180m° \le x < 180(m+1)°.$$

The excluded denominator cases do not remove any of these intersections, because when $\sin x = 1$ or $\sin 2x = 1$, the equation $|\cos x| = |\cos 2x|$ is not satisfied.

Therefore, in every $180°$ block there are exactly 3 solutions. In the interval

$$0° \le x < 900°,$$

there are 5 complete blocks, so there are

$$5 \cdot 3 = 15$$

solutions.

The next block starts at $900°$, and at this point both graphs have value 1, so $x = 900°$ is the next solution. Since the interval is $0° \le x < p°$, we need $p > 900$ to include this solution.

Therefore the minimum integer value of p is 901.