JZ Mock Set B Paper 1 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

Last updated: 5 September 2026 at 23:13

Spotted an error? Please email jzmaths@hotmail.com.

Question 1

The equation

$$x^{\log_2 x} = \frac{x^3}{4}$$

has exactly two solutions for $x > 0$. Find their product.

A 2

B 4

C 6

D 8

E 16

F 20

Solution 1

Answer: D

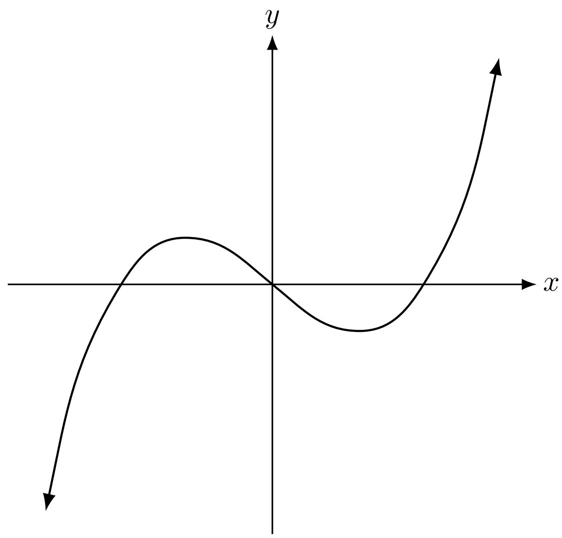
Take logarithms base 2 of both sides. The left side becomes $(\log_2 x)(\log_2 x)$, and the right becomes $3\log_2 x - 2$. Writing $u = \log_2 x$,

$$u^2 = 3u - 2, \qquad \text{so} \qquad u^2 - 3u + 2 = 0,$$

giving $u = 1$ or $u = 2$, that is $x = 2$ or $x = 4$. Their product is $2^{u_1} \cdot 2^{u_2} = 2^{u_1+u_2} = 2^3 = 8$.

Question 2

The curve $y = x^3 - 4x$ is shown below.



What is the total area enclosed between the curve, the x-axis and the lines $x = -3$ and $x = 3$?

A 0

B $\frac{9}{2}$

C 8

D $\frac{25}{2}$

E $\frac{41}{2}$

F $\frac{33}{2}$

Solution 2

Answer: E

Due to symmetry, we can just do:

$$2 \times \left(\int_0^2 -y dx + \int_2^3 y dx \right) = 2 \times \left(4 + \frac{25}{4} \right) = \frac{41}{2}.$$

Question 3

Evaluate

$$\sum_{n=0}^{20} \log_{\frac{1}{2}}(8^{-n}).$$

A 630

B $231 \log_{\frac{1}{2}} 8$

C $-231 \log_{\frac{1}{2}} 8$

D -693

E $-210 \log_2 8$

F 693

G $210 \log_{\frac{1}{2}} 8$

H $231 \log_2 8$

Solution 3

Answer: A

Pull out the constant: $\log_{\frac{1}{2}}(8^{-n}) = -n \log_{\frac{1}{2}}(8)$. Since $\log_{\frac{1}{2}}(8) = \log_{\frac{1}{2}}(2^3) = -3$, this simplifies to $3n$. Hence

$$\sum_{n=0}^{20} \log_{\frac{1}{2}}(8^{-n}) = \sum_{n=0}^{20} 3n = 3 \cdot \frac{20 \cdot 21}{2} = 630.$$

Question 4

A circle has equation
$$x^2 + y^2 - 4x + 6y - 3 = 0.$$

The circle is first translated by 5 units in the positive x-direction, then reflected in the line $y = x$, then reflected in the x-axis, and finally enlarged by a scale factor of 3 about the origin. The equation of the final circle is

A $(x + 9)^2 + (y + 21)^2 = 144$

B $(x + 9)^2 + (y + 21)^2 = 48$

C $(x + 9)^2 + (y + 21)^2 = 16$

D $(x + 9)^2 + (y - 21)^2 = 144$

E $(x - 9)^2 + (y + 21)^2 = 144$

F $(x + 21)^2 + (y + 9)^2 = 144$

G $(x + 3)^2 + (y + 7)^2 = 144$

H $(x + 9)^2 + (y + 21)^2 = 90$

Solution 4

Answer: A

Completing the squares:

$$x^2 - 4x + y^2 + 6y - 3 = 0 \implies (x - 2)^2 - 4 + (y + 3)^2 - 9 - 3 = 0 \implies (x - 2)^2 + (y + 3)^2 = 16.$$

So the original centre is $(2, -3)$ and the radius is 4.

Apply the transformations to the centre (each is a rigid motion except the last). (1) Translate by $+5$ in x: centre becomes $(7, -3)$, radius still 4. (2) Reflect in $y = x$ (swap coordinates): centre becomes $(-3, 7)$, radius still 4. (3) Reflect in the x-axis (negate y): centre becomes $(-3, -7)$, radius still 4. (4) Enlarge by scale factor 3 about the origin (multiply each coordinate by 3, multiply radius by 3): centre becomes $(-9, -21)$, radius becomes 12.

The final equation is $(x - (-9))^2 + (y - (-21))^2 = 12^2$, i.e. $(x + 9)^2 + (y + 21)^2 = 144$.

Question 5

The function f is continuous for all real x, and

$$I = \int_1^5 f(x)\,dx.$$

Which one of the following is **necessarily** equal to I?

A

$$\int_{-2}^{2} \left[f(x+3) + x^2\right] dx$$

B

$$\int_{-2}^{2} \left[f(x+3) + x^3\right] dx$$

C

$$\int_{-2}^{2} \left[f(x-3) + x^3\right] dx$$

D

$$\int_{-2}^{2} \left[f(x-3) + x^2\right] dx$$

E

$$\int_{-1}^{3} f(x-2)dx$$

F

$$\int_{-1}^{3} \left[f(x+2) + x^2\right] dx$$

Solution 5

Answer: B

The graph of $y = f(x+3)$ is the graph of $y = f(x)$ translated 3 units to the left. Therefore, the area under $y = f(x+3)$ from $x = -2$ to $x = 2$ is equal to the area under $y = f(x)$ from $x = 1$ to $x = 5$. So

$$\int_{-2}^{2} f(x+3)\,dx = I.$$

Also, the graph of $y = x^3$ has rotational symmetry about the origin, so the signed area from $x = -2$ to $x = 2$ is 0. Hence

$$\int_{-2}^{2} x^3\,dx = 0.$$

Therefore

$$\int_{-2}^{2} \left[f(x+3) + x^3\right]\, dx = I + 0 = I.$$

The other options either add a positive x^2 term, or use a horizontal translation of f over the wrong interval.

Question 6

Which of the following represents the set of x satisfying: $x^2 > x$ and $1 < 17 - 4x$.

A $(x-1)(x-4) < 0$

B $x^2 - 5x + 4 > 0$

C $\dfrac{3}{x-1} > x - 3$

D $x^3 - 5x^2 + 4x > 0$

E $x(x-1) < 0$

F $x^2 - x > 0$

Solution 6

Answer: C

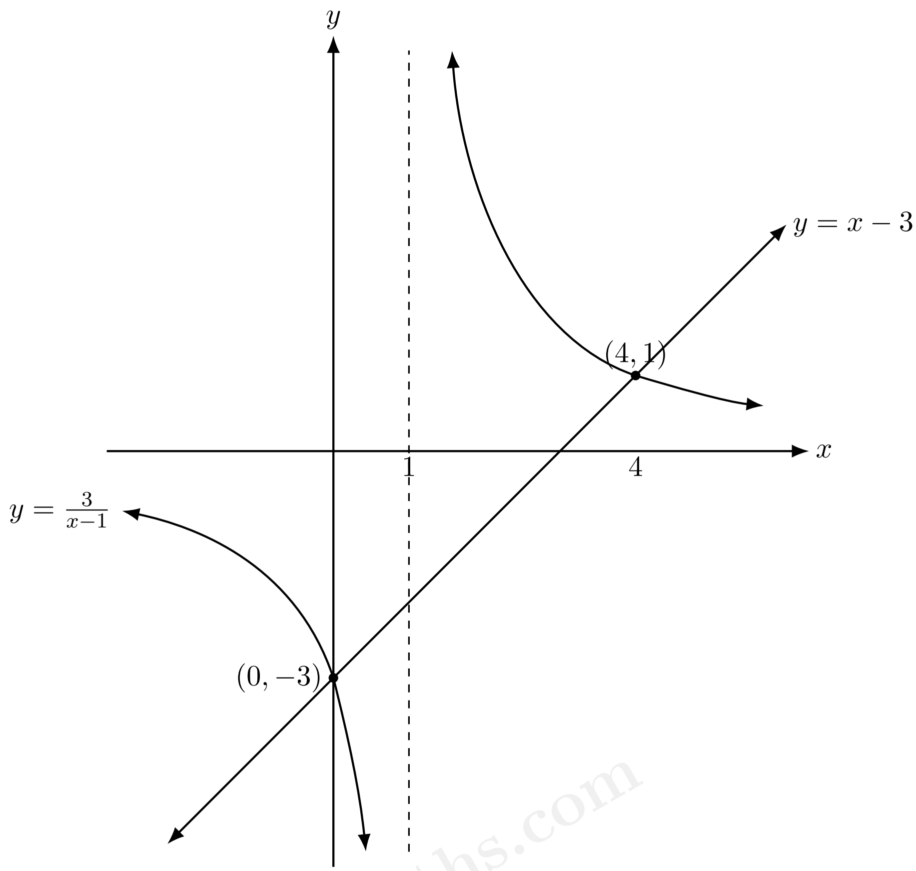
First, find the solution for the simultaneous inequalities. From $x^2 > x$: $x(x-1) > 0$, so $x < 0$ or $x > 1$. From $1 < 17 - 4x$: $x < 4$. Intersection is $(-\infty, 0) \cup (1, 4)$.

This cannot be achieved by any quadratic inequality, therefore the only possible options are **C** and **D**.

C: we could use the graph method for this. Sketch $y = \dfrac{3}{x-1}$ and $y = x - 3$ on the same axes. Intersections satisfy

$$\frac{3}{x-1} = x - 3 \Leftrightarrow x = 0, 4,$$

so the curves meet at $(0, -3)$ and $(4, 1)$.



Using the intersection points and asymptote at $x = 1$, we can observe it gives the correct set of x of $(-\infty, 0) \cup (1, 4)$, so it is the correct option.

For completeness, let's check **D:** $x^3 - 5x^2 + 4x = x(x-1)(x-4) > 0$ gives $0 < x < 1$ or $x > 4$. Wrong sign; the correct cubic representation is $x(x-1)(x-4) < 0$.

Question 7

Given $-3 < x < 3$, find the total length of the intervals in which

$$\sqrt{(x-2)^2} \leq x^2 - 3x.$$

A $4 - \sqrt{3}$

B 3

C $5 - \sqrt{2}$

D $\sqrt{2} + 1$

E $\sqrt{3} + 2$

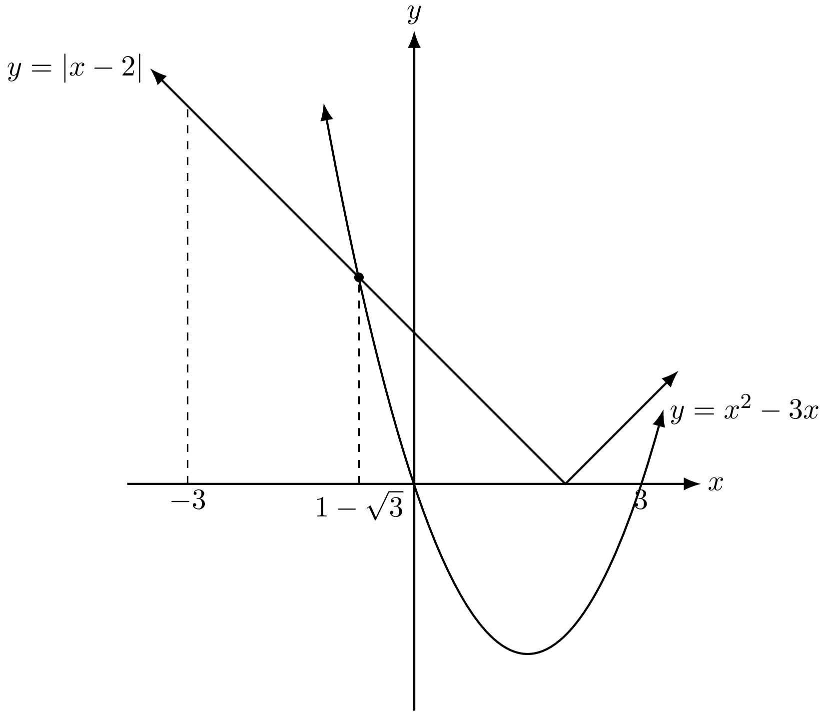
F $6 - 2\sqrt{3}$

Solution 7

Answer: A

Yes here is another example of the infamous $\sqrt{x^2} \neq x$ but is equal to $|x|$ question, by the time you are done with my mocks, this will be hard to forget!

Since $\sqrt{(x-2)^2} = |x-2|$, we need $|x-2| \leq x^2 - 3x$. The two graphs are shown below.



From the graphs, it becomes immediately clear that we only need to find the intersection between the left arm $2-x$ and x^2-3x. Solving $2-x = x^2-3x$, gives $x = 1 \pm \sqrt{3}$, so the relevant solution is $x = 1-\sqrt{3}$. Note that we don't need to find the intersection of the right arm of V with the quadratic, because that clearly that intersection occurs for $x > 3$.

Therefore, using the graphs and the intersection at $x = 1-\sqrt{3}$, we deduce the interval is $-3 < x \leq 1-\sqrt{3}$, and its length is $(1-\sqrt{3}) - (-3) = 4-\sqrt{3}$.

Question 8

Evaluate the following integral.

$$\int_{-2}^{2} x^2|1-x^2|\,dx$$

A 6

B 2

C 4

D 8

E 0

F $\dfrac{13}{2}$

Solution 8

Answer: D

The integrand $x^2|1-x^2|$ is an even function, so $\int_{-2}^{2} x^2|1-x^2|\,dx = 2\int_0^2 x^2|1-x^2|\,dx$. Split at $x=1$, where $1-x^2$ changes sign.

On $[0,1]$: $|1-x^2| = 1-x^2$, so the integrand is x^2-x^4.

$$\int_0^1 (x^2-x^4)\,dx = \left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1 = \frac{1}{3}-\frac{1}{5} = \frac{2}{15}.$$

On $[1,2]$: $|1-x^2| = x^2-1$, so the integrand is x^4-x^2.

$$\int_1^2 (x^4-x^2)\,dx = \left[\frac{x^5}{5}-\frac{x^3}{3}\right]_1^2 = \frac{31}{5}-\frac{7}{3} = \frac{58}{15}.$$

Summing and doubling: $2\left(\frac{2}{15}+\frac{58}{15}\right) = 2\cdot 4 = 8.$

Question 9

Let a, b and c be non-zero integers. The expression

$$\frac{6^{a+b+c} \cdot 10^{a-b-c}}{15^{a-b+c}}$$

is a positive integer if which of the following is true?

A $a > 0$, $b > 0$ and $c < 0$

B $a > 0$, $b > 0$ and $c > 0$

C $a > 0$, $b < 0$ and $c < 0$

D $a < 0$, $b > 0$ and $c < 0$

E $a < 0$, $b < 0$ and $c > 0$

F $a < 0$, $b < 0$ and $c < 0$

G $a > 0$ and $c < 0$ (no condition on b)

H $c < 0$ only (no conditions on a or b)

Solution 9

Answer: A

Disclaimer: this is a variation of a TMUA specimen question, a type of question which I felt is very useful to practice more of, so here it is!

Factor each base into primes: $6 = 2 \cdot 3$, $10 = 2 \cdot 5$, $15 = 3 \cdot 5$.

The exponent of each prime in the expression is:

$$\begin{aligned}
\text{prime } 2 &: (a+b+c) + (a-b-c) = 2a, \\
\text{prime } 3 &: (a+b+c) - (a-b+c) = 2b, \\
\text{prime } 5 &: (a-b-c) - (a-b+c) = -2c.
\end{aligned}$$

So the expression equals $2^{2a} \cdot 3^{2b} \cdot 5^{-2c}$.

This is a positive integer if and only if every prime exponent is non-negative, i.e. $2a \geq 0$, $2b \geq 0$ and $-2c \geq 0$. Since a, b, c are non-zero integers, this is equivalent to $a > 0$, $b > 0$ and $c < 0$.

Question 10

Which of the following integrals has the **greatest** value?

A

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^{\frac{1}{3}} x)\, dx$$

B

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos(\frac{\pi}{2} - x)|\, dx$$

C

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 x)\, dx$$

D

$$\int_0^{\pi} \sqrt{|\sin\left(x - \frac{\pi}{2}\right)|}\, dx$$

E

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \cos^2 x}\, dx$$

Solution 10

Answer: D

Option A is 0 since $\sin^{\frac{1}{3}} x$ is an odd function.

Option B: $|\cos(\frac{\pi}{2} - x)| = |\sin x|$, therefore value of B is 2 times the area under the graph of $\sin x$ between 0 to $\frac{\pi}{2}$.

Option E: $\sqrt{1 - \cos^2 x} = \sqrt{\sin^2 x} = |\sin x|$, so B and E have the same area, and so are both eliminated - there cannot be two correct options!

Option C: $\sin^2 x < |\sin x|$ since $-1 \leq \sin x \leq 1$, so C has an area less than B or E.

Therefore it must be D.

Let's check it anyway. $\sqrt{|\sin\left(x - \frac{\pi}{2}\right)|} = \sqrt{|\cos x|}$, therefore value of D is 2 times the area under the graph of $\sqrt{\cos x}$ between 0 to $\frac{\pi}{2}$, this is the same as that of $\sqrt{\sin x}$, which is greater than $\sin x$ since $\sqrt{a} \geq a$ if a is between 0 and 1.

Question 11

Given

$$a_n = \sin\left(\frac{2\pi}{3}n\right) + \cos\left(\frac{\pi}{6}(1+4n)\right)$$

and

$$S = \sum_{n=1}^{k} a_n.$$

For how many values of positive integer k such that $1 \le k \le 100$, is $S = 0$.

A 0

B 33

C 34

D 66

E 67

F 100

Solution 11

Answer: E

As n increases by 1, the angle in $\sin\left(\frac{2\pi}{3}n\right)$ increases by $\frac{2\pi}{3}$. Hence this term has period 3.

For $n = 1, 2, 3$, its values are

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \qquad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \qquad \sin(2\pi) = 0.$$

Similarly, as n increases by 1, the angle in $\cos\left(\frac{\pi}{6}(1+4n)\right)$ also increases by $\frac{2\pi}{3}$. Hence this term also has period 3.

For $n = 1, 2, 3$, its values are

$$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \qquad \cos\left(\frac{3\pi}{2}\right) = 0, \qquad \cos\left(\frac{13\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Therefore the sequence a_n repeats every 3 terms, with values obtained by adding the corresponding values of the cos and sin terms:

$$0, \quad -\frac{\sqrt{3}}{2}, \quad \frac{\sqrt{3}}{2}.$$

So each complete block of 3 terms has sum 0.

The **running sums**: a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$ are therefore:

$$0, \quad -\frac{\sqrt{3}}{2}, \quad 0.$$

So $S = 0$ when k is either the first or third position in one of these repeated blocks of 3, or we can say is when k is 1 or 0 mod 3.

From 1 to 99, there are 33 complete blocks, giving 2 suitable values of k from each block. This gives

$$33 \cdot 2 = 66$$

values. The remaining value $k = 100$ is the first position in the next block, so it gives one more suitable value.

Hence the total number of values is $66 + 1 = 67$.

Question 12

In the expansion of $(1+x)^n$, the coefficients of three consecutive terms are in the ratio $1:2:3$. Find n.

A 9

B 10

C 11

D 12

E 13

F 14

G 15

Solution 12

Answer: F

Let the three consecutive terms involve x^{r-1}, x^r, x^{r+1}, so that

$$\binom{n}{r-1} : \binom{n}{r} : \binom{n}{r+1} = 1:2:3.$$

Using

$$\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n-r+1}{r},$$

the two ratios give

$$\frac{n-r+1}{r} = 2 \qquad \text{and} \qquad \frac{n-r}{r+1} = \frac{3}{2}.$$

The first rearranges to $n+1=3r$ and the second to $2n=5r+3$. Solving simultaneously gives $r=5$ and $n=14$.

Question 13

Three sectors of circles are similar (they share the same central angle) and their radii form an arithmetic progression. The smallest sector has arc length 6. The area of the middle sector exceeds the area of the smallest sector by 21. The area of the largest sector exceeds the area of the middle sector by 27. Find the positive difference between the perimeters of the largest and smallest sectors.

A 7

B 8

C 10.5

D 12

E 14

F 16

G 18

Solution 13

Answer: F

Let the common central angle be θ and let the radii be r, $r+s$, $r+2s$ for some $s>0$. From the smallest arc, $r\theta = 6$. The area of a sector is $\frac{1}{2}r^2\theta$, so:

$\frac{1}{2}\big((r+s)^2 - r^2\big)\theta = 21 \;\Rightarrow\; (2rs+s^2)\theta = 42.$

$\frac{1}{2}\big((r+2s)^2 - (r+s)^2\big)\theta = 27 \;\Rightarrow\; (2rs+3s^2)\theta = 54.$

Subtracting gives $2s^2\theta = 12$, so $s^2\theta = 6$. Then $(2rs+s^2)\theta = 42$ becomes $2s(r\theta) + s^2\theta = 42$, i.e. $12s + 6 = 42$, so $s = 3$. Hence $\theta = \frac{6}{s^2} = \frac{2}{3}$ and $r = \frac{6}{\theta} = 9$.

The radii are 9, 12, 15 and the arc lengths are $r\theta$, $(r+s)\theta$, $(r+2s)\theta = 6, 8, 10$. The perimeter of a sector equals $2r + r\theta$, giving perimeters 24, 32, 40. The positive difference between the largest and smallest perimeters is $40 - 24 = 16$.

Question 14

How many solutions does the equation

$$\sqrt{1-\cos^2(2x)} = \left(x - \frac{3\pi}{4}\right)\cos(2x)$$

have for x in the interval $0 < x < \pi$?

A 0

B 1

C 2

D 3

E 4

F 5

Solution 14

Answer: D

Since $\sqrt{1-\cos^2(2x)} = |\sin(2x)|$,the equation becomes

$$|\sin(2x)| = \left(x - \frac{3\pi}{4}\right)\cos(2x).$$

Values of x for which $\sin 2x$ and $\cos 2x$ equal to zero do not coincide at all. Therefore we can ignore any x which make $\cos 2x$ zero, and hence we can divide by $\cos 2x$. Another way to say this is that values of x for which $\cos 2x$ is zero can be checked and verified as not solutions to the equation, so we can immediately exclude them first, then $\cos 2x$ is now non-zero. Therefore we may divide by $\cos(2x)$, giving

$$\frac{|\sin(2x)|}{\cos(2x)} = x - \frac{3\pi}{4}.$$

For $0 < x < \frac{\pi}{2}$, we have $\sin(2x) > 0$, so

$$\frac{|\sin(2x)|}{\cos(2x)} = \tan(2x).$$

For $\frac{\pi}{2} < x < \pi$, we have $\sin(2x) < 0$, so

$$\frac{|\sin(2x)|}{\cos(2x)} = -\tan(2x).$$

Now sketch

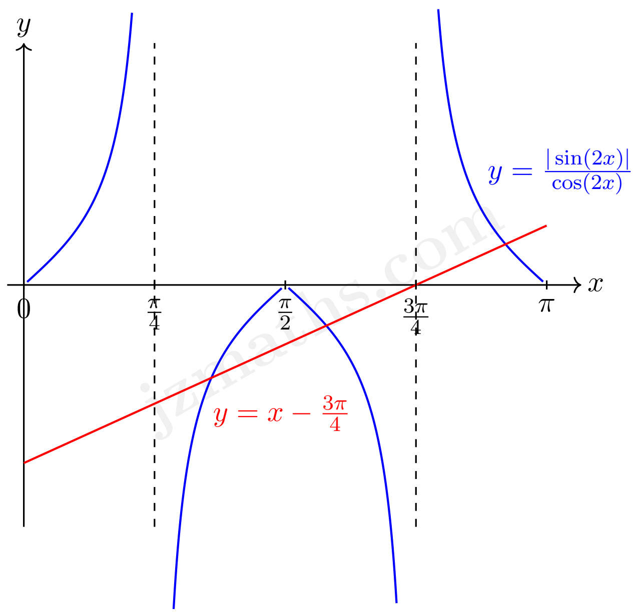
$$y = \frac{|\sin(2x)|}{\cos(2x)}$$

as a piecewise function: $\tan 2x$ between 0 to $\pi/2$, and $-\tan 2x$ between $\pi/2$ to π.

Then sketch

$$y = x - \frac{3\pi}{4}$$

on the same axes.



It becomes immediately clear there are 3 intersection points, and so 3 solutions.

Question 15

The circles C_1 and C_2 are defined by the equations

$$C_1: \ (x-2)^2 + (y-3)^2 = 9$$

$$C_2: \ (x-11)^2 + (y-15)^2 = 16$$

A common tangent to C_1 and C_2 touches C_1 at X and C_2 at Y. Find the sum of the squares of all distinct possible lengths of XY.

A 1

B 49

C 176

D 224

E 400

F 450

Solution 15

Answer: E

Let A and B be the centres C_1 and C_2 respectively, then the distance between the centres is

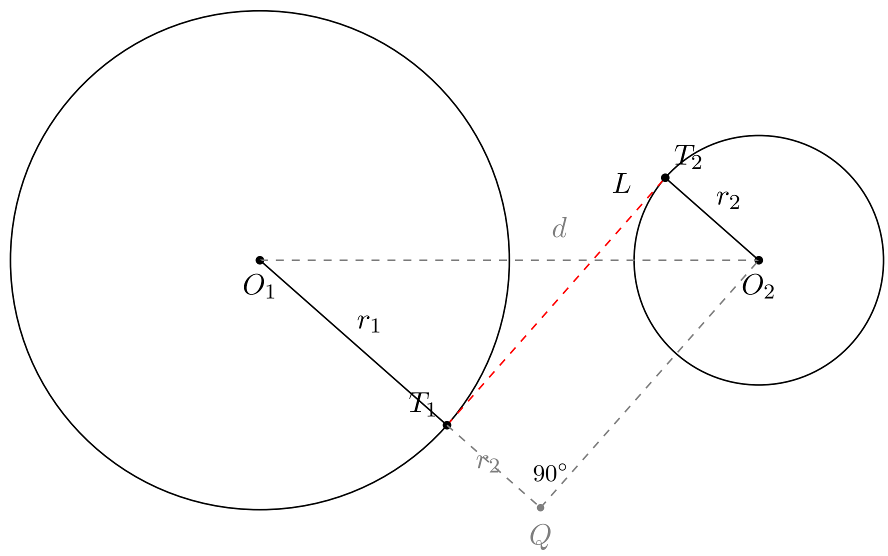
$$AB = \sqrt{(11-2)^2 + (15-3)^2 = 9^2 + 12^2} = 15.$$

There are two types of common tangents: external common tangents and internal common tangents.

For an external common tangent, where the circles are on the same side of the tangent line, the radii to the points of contact are parallel. The difference between the radii is $4 - 3 = 1$. So the length between the two points of contact is by Pythagorus:

$$XY^2 = AB^2 - 1^2 = 225 - 1 = 224.$$

For an internal common tangent, the radii to the points of contact point in opposite directions. The relevant distance is the sum of the radii $3 + 4 = 7$, which together with XY and AB can form a right angled triangle. There are a couple of ways this can be shown.



So the length between the two points of contact is given by Pythagorus:

$$XY^2 = AB^2 - (4+3)^2 = 225 - 49 = 176.$$

Therefore the sum of distinct values of XY^2 is

$$224 + 176 = 400.$$

Question 16

An arithmetic sequence (a_n) and a convergent geometric sequence (g_n) are combined to form a new sequence (T_n), where $T_n = a_n + g_n$. Given that

$$T_1 = 3, \quad T_2 = 2, \quad T_3 = \frac{3}{2}, \quad T_4 = \frac{9}{8},$$

find the sum to infinity of the geometric sequence (g_n).

A $\frac{8}{9}$

B $\frac{32}{27}$

C $\frac{4}{3}$

D $\frac{3}{2}$

E $\frac{27}{8}$

F 4

Solution 16

Answer: B

The nth terms of the two sequence be $p + (n-1)d$ and qr^{n-1}, so

$$T_n = p + (n-1)d + qr^{n-1}.$$

Form the first differences $\Delta_n = T_{n+1} - T_n$:

$$\begin{aligned}
\Delta_1 &= d + q(r-1) = 2 - 3 = -1, \\
\Delta_2 &= d + qr(r-1) = \frac{3}{2} - 2 = -\frac{1}{2}, \\
\Delta_3 &= d + qr^2(r-1) = \frac{9}{8} - \frac{3}{2} = -\frac{3}{8}.
\end{aligned}$$

The constant d cancels in second differences:

$$\begin{aligned}
\Delta_2 - \Delta_1 &= q(r-1)(r-1) = q(r-1)^2 = \frac{1}{2}, \\
\Delta_3 - \Delta_2 &= q(r^2 - r)(r-1) = qr(r-1)^2 = \frac{1}{8}.
\end{aligned}$$

Dividing the second by the first gives $r = \frac{1}{4}$, which satisfies $|r| < 1$ as required.

Then $q(r-1)^2 = q \cdot \frac{9}{16} = \frac{1}{2}$, so $q = \frac{8}{9}$. Hence

$$S_\infty = \frac{q}{1-r} = \frac{8/9}{3/4} = \frac{32}{27}.$$

Question 17

Evaluate

$$\int_0^{\pi/2} \frac{1}{1+\tan x}\, dx.$$

You may find the identity

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan x}$$

useful.

A $\frac{\pi}{2}$

B $\frac{\pi}{4}$

C $\frac{\pi}{2\sqrt{2}}$

D $\frac{\pi}{4\sqrt{2}}$

E $\frac{\pi}{3}$

F $\frac{\pi}{6}$

Solution 17

Answer: B

Let

$$I = \int_0^{\pi/2} \frac{1}{1+\tan x}\, dx.$$

In order to use the given identity, we need to introduce $\tan(\pi/2 - x)$. The best way to achieve this is through a reflection along $x = \pi/4$. See my remark at the end for more details on this. This changes x into $\frac{\pi}{2} - x$, but leaves the interval $0 \leq x \leq \frac{\pi}{2}$ unchanged. Hence

$$I = \int_0^{\pi/2} \frac{1}{1+\tan\left(\frac{\pi}{2} - x\right)}\, dx.$$

Now using the identity

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan x},$$

we get

$$I = \int_0^{\pi/2} \frac{1}{1+\frac{1}{\tan x}}\, dx.$$

So

$$I = \int_0^{\pi/2} \frac{\tan x}{1 + \tan x}\, dx.$$

Adding this to the original expression for I gives

$$2I = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}.$$

Therefore

$$I = \frac{\pi}{4}.$$

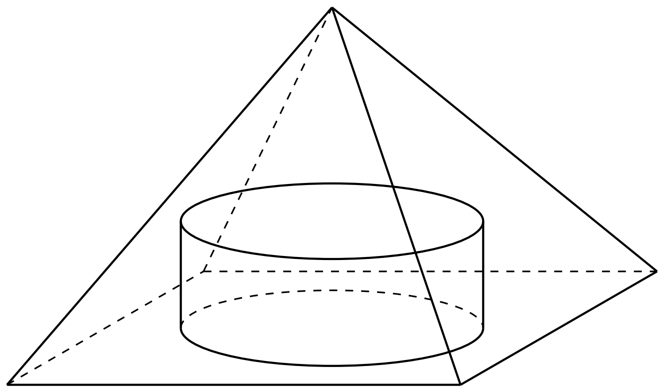
Remark: The solution leans heavily on the fact that in general

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx.$$

This can be understood as a reflection in the vertical line $x = \frac{a+b}{2}$, this is the midpoint of the interval a to b. Under this reflection, a point with coordinate x is sent to $a + b - x$. The interval $a \le x \le b$ is unchanged by this reflection, so the total area under the graph is unchanged. Hence replacing $f(x)$ by $f(a+b-x)$ gives the same integral.

Question 18

A right circular cylinder is placed inside a regular square-based pyramid. One circular face of the cylinder lies in the square base of the pyramid, and the other circular face is parallel to it. All edges of the pyramid have the same length.



What is the largest possible value of

$$\frac{\text{volume of the cylinder}}{\text{volume of the pyramid}}?$$

A $\frac{\pi}{9}$

B $\frac{\pi}{10}$

C $\frac{\sqrt{2}\pi}{9}$

D $\frac{\pi}{12}$

E $\frac{2\sqrt{2}\pi}{27}$

Solution 18

Answer: A

Since the required answer is a proportion, we may choose a convenient edge length. Let every edge of the pyramid have length $\sqrt{2}$, we will see shortly that this leads to a most convenient value for the height of the pyramid.

By Pythagorus, the distance from the centre of the square base to a vertex is 1. Since the sloping edge also has length $\sqrt{2}$, the height H of the pyramid satisfies

$$H^2 + 1^2 = (\sqrt{2})^2 \quad \Rightarrow \quad H = 1.$$

At height h above the base, the horizontal cross-section of the pyramid is a square similar to the base square. Since the total height of the pyramid is 1, the smaller pyramid above this cross-section has height $1-h$. By similar pyramids, the ratio of corresponding lengths in the original pyramid and this smaller pyramid is

$$1 : (1-h).$$

Therefore the side length of the square cross-section is scaled by a factor of $1-h$ compared with the base square. Since the base square has side length $\sqrt{2}$, the side length of the square cross-section is

$$\sqrt{2}(1-h).$$

So the radius of the largest circle that fits inside this square is

$$r = \frac{\sqrt{2}}{2}(1-h).$$

Hence the volume of the cylinder is

$$V = \pi r^2 h = \frac{\pi}{2} h(1-h)^2.$$

So we need to maximise $h(1-h)^2$. Now

$$h(1-h)^2 = h - 2h^2 + h^3.$$

Differentiating gives

$$1 - 4h + 3h^2 = (1-h)(1-3h).$$

The maximum occurs at

$$h = \frac{1}{3}.$$

Therefore

$$r = \frac{\sqrt{2}}{2}\left(1 - \frac{1}{3}\right) = \frac{\sqrt{2}}{3}.$$

So the largest cylinder has volume

$$V_{\text{cyl}} = \pi \left(\frac{\sqrt{2}}{3}\right)^2 \frac{1}{3} = \frac{2\pi}{27}.$$

The volume of the pyramid is

$$V_{\text{pyr}} = \frac{1}{3} \cdot 2 \cdot 1 = \frac{2}{3}.$$

Hence the required proportion is

$$\frac{V_{\text{cyl}}}{V_{\text{pyr}}} = \frac{\frac{2\pi}{27}}{\frac{2}{3}} = \frac{\pi}{9}.$$

Remark: As a general rule of thumb, I try not to make these questions too long, though this one may be an exception. However, I think the length is justified here, since the question tests several useful ideas: recognising the role of similar pyramids, setting up the cylinder volume, and choosing a convenient edge length to simplify the calculation. There did not seem to be an obvious way to shorten the question meaningfully without losing these essential learning points. So as a practice mock question, it serves its purpose well.

Question 19

Let p be a real number, and let n denote the number of points of intersection of the curves $y = |x^3 - p^3|$ and $y = p^3|x-1|$. n **cannot** take which of the following values?

A 0

B 1

C 4

D 3

E 2

Solution 19

Answer: E

Setting the curves equal: $|x^3 - p^3| = p^3|x-1|$.

Case $p < 0$. Then $p^3 < 0$, so LHS $\geq 0 \geq$ RHS. Equality forces both sides to vanish, requiring $x = p$ and $x = 1$ simultaneously, which is impossible. Hence $n = 0$.

Case $p = 0$. The equation reduces to $|x|^3 = 0$, giving the single root $x = 0$. Hence $n = 1$.

Case $p > 0$. There are a couple of ways to analyse this, and sketching graphs is a great method, since both curves are easy to sketch for $p > 0$.

Now, the absolute value causes the graph of $x^3 - p^3$ to be reflected where it would be negative, with the turning point at $x = p$. Similarly, the graph of $x - 1$ is reflected where it would be negative, with the corner at $x = 1$. This indicates that we should examine three sub-cases: $0 < p < 1$, $p = 1$ and $1 < p$.

For $0 < p < 1$, if you sketch both graphs, for example with $p = 1/2$, you will observe that $-(x^3 - 1/8)$ and $1/8(-x+1)$ intersect 3 times, and $(x^3 - 1/8)$ and $1/8(-x+1)$ intersect once, so $n = 4$ in this case. The same qualitative shape holds for all $0 < p < 1$, so for all these cases, $n = 4$.

For $p = 1$, if you sketch both graphs, you will observe that $-(x^3 - 1)$ and $(-x+1)$ intersect 3 times, and the two reflection points coincide at $x = 1$. So altogether in this case, there are still 3 intersections, and hence $n = 3$.

For $1 < p$, if you sketch both graphs, for example with $p = 2$, you will observe that $-(x^3 - 8)$ and $8(-x+1)$ intersect 2 times, $-(x^3 - 8)$ and $8(x-1)$ intersect once, and $(x^3 - 8)$ and $8(x-1)$ intersect once, so $n = 4$ in this case too. The same qualitative shape holds for all $1 < p$, so for all these cases, $n = 4$.

Therefore the only value in the options n cannot take is 2.

Question 20

Let $y = t^{\sin 2x} - 2\,t^{-\sin 2x}$, where $t > 0$ and x is real. For what values of t, is the product between the maximum value of y and the minimum value of y positive.

A $\quad 0 < t < \frac{1}{\sqrt{2}}$

B $\quad t > \sqrt{2}$

C $\quad \frac{1}{\sqrt{2}} < t < \sqrt{2}$

D $\quad \frac{1}{\sqrt{2}} \leq t \leq \sqrt{2}$

E $\quad 0 < t < \sqrt{2}$

F $\quad 1 \geq t < \sqrt{2}$

Solution 20

Answer: C

Let $u = t^{\sin 2x}$. Then $y = u - 2/u$.

For $u > 0$, the expression $u - \frac{2}{u}$ increases as u increases, because

$$\frac{d}{du}\left(u - \frac{2}{u}\right) = 1 + \frac{2}{u^2} > 0.$$

Therefore, to find the maximum and minimum values of y, we only need to determine the maximum and minimum values of u. For this, we distinguish between $0 < t < 1$ and $t \geq 1$. When $0 < t < 1$, the function $y = t^x$ is decreasing; for example, $y = 0.5^x$ is the same as $y = 2^{-x}$. When $t > 1$, the function is increasing; for example, $y = 3^x$ is increasing. When $t = 1$, the function is simply constant.

Case 1: $t \geq 1$.

Since $-1 \leq \sin 2x \leq 1$, and t^z increases with z when $t > 1$ and is constant when $t = 1$, we have $\frac{1}{t} \leq u \leq t$. Hence the maximum value of y is $t - \frac{2}{t}$, and the minimum value is $\frac{1}{t} - 2t$.

Since $t \geq 1$, we have $\frac{1}{t} - 2t < 0$. Indeed, multiplying by $t > 0$, this is equivalent to $\frac{1}{2} < t^2$, which is certainly true. Therefore, the product of the maximum and minimum values is positive exactly when $t - \frac{2}{t} < 0$. Multiplying by $t > 0$ gives $t^2 < 2$, so $t < \sqrt{2}$. Thus, in this case,

$$1 \leq t < \sqrt{2}.$$

Case 2: $0 < t < 1$.

Now t^z decreases as z increases, so $t \le u \le \frac{1}{t}$. Hence the maximum value of y is $\frac{1}{t} - 2t$, and the minimum value is $t - \frac{2}{t}$.

Since $0 < t < 1$, we have $t - \frac{2}{t} < 0$. Indeed, multiplying by $t > 0$, this is equivalent to $t^2 < 2$, which is certainly true. Therefore, the product of the maximum and minimum values is positive exactly when $\frac{1}{t} - 2t < 0$. Multiplying by $t > 0$ gives $1 - 2t^2 < 0$, so $t > \frac{1}{\sqrt{2}}$. Thus, in this case,

$$\frac{1}{\sqrt{2}} < t < 1.$$

Combining the two cases gives

$$\frac{1}{\sqrt{2}} < t < \sqrt{2}.$$

Remark: Sketching the graph of $y = x - 2/x$ makes both of these sign checks much easier to see. The graph lies below the x-axis for $0 < x < \sqrt{2}$. Hence, if $0 < t < 1$, then $t - 2/t < 0$. Similarly, if $t \ge 1$, then $0 < 1/t \le 1 < \sqrt{2}$, so $1/t - 2/(1/t) = 1/t - 2t < 0$.