JZ Mock Set B Paper 2 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

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Question 1

It is a fact that

$$1729 = 1^3 + 12^3 = 9^3 + 10^3 = 7 \times 13 \times 19.$$

Consider the following four statements.

(1) Every positive integer that can be written as a sum of two positive cubes in two different ways is prime.

(2) If a prime $p > 5$ can be written as $p = a^3 + b^3$ for some positive integers a, b, then p has remainder of 1 when divided by 6.

(3) No product of three distinct odd primes is one more than a perfect cube.

(4) Every positive integer of the form $a^3 + b^3$, where a and b are positive integers, has prime factors all of which are at least 3.

The fact above can be used to provide a counterexample to which of these statements?

A none of them

B 1 only

C 3 only

D 1 and 3 only

E 2 and 3 only

F 1, 2 and 3 only

G 1, 3 and 4 only

H 1, 2, 3 and 4

Solution 1

Answer: D

(1) says that every positive integer which can be written as a sum of two positive cubes in two different ways is prime.

But 1729 can be written as a sum of two positive cubes in two different ways:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Also, $1729 = 7 \times 13 \times 19$, so 1729 is not prime. Therefore the fact gives a counterexample to statement (1).

(2) is about a prime $p > 5$. The number 1729 is not prime, statement (2) simply does not apply to 1729, therefore it is not a counterexample of (2).

(3) says that no product of three distinct odd primes is one more than a perfect cube. 1729 is product of three distinct primes, so it qualifies, and it **is** one more than a cube: $1729 = 12^3 + 1$, therefore it is a counterexample to (3).

(4) says that every positive integer of the form $a^3 + b^3$ has prime factors all of which are at least 3. For 1729, the prime factors are 7, 13 and 19, all of which are at least 3. So the fact does not give a counterexample to statement (4).

Therefore the fact can be used to provide counterexamples to statements **1 and 3 only**.

Question 2

A student is asked to solve the equation

$$\frac{\sqrt{x^3 + x^2 - x} + 2}{x + 2} = 1.$$

The student writes:

Step (1): Multiply both sides by $x + 2$ to obtain

$$\sqrt{x^3 + x^2 - x} + 2 = x + 2.$$

Step (2): Subtract both sides by 2 to get

$$\sqrt{x^3 + x^2 - x} = x.$$

Step (3): Square both sides to obtain

$$x^3 + x^2 - x = x^2.$$

Step (4): Rearrange to obtain

$$x^3 - x = 0.$$

Step (5): Factorise to obtain

$$x(x - 1)(x + 1) = 0.$$

Step (6): Deduce

$$x = -1, 0, 1.$$

Which one of the following statements is correct?

A The final answers for x are wrong, and the first error appears in Step (1).

B The final answers for x are wrong, and the first error appears in Step (2).

C The final answers for x are wrong, and the first error appears in Step (3).

D The final answers for x are wrong, and the first error appears in Step (4).

E The final answers for x are wrong, and the first error appears in Step (5).

F The final answers for x are wrong, and the first error appears in Step (6).

G The final answers for x are correct, and there is no error in the student's working.

Solution 2

Answer: C

The original equation has denominator $x+2$, so $x \neq -2$. Therefore multiplying both sides by $x+2$ in Step (1) is valid.

Step (2) is also valid, giving
$$\sqrt{x^3 + x^2 - x} = x.$$

However, this equation also tells us that $x \geq 0$, since the left hand side is a square root.

In Step (3), squaring both sides gives

$$x^3 + x^2 - x = x^2,$$

which simplifies to
$$x^3 - x = 0.$$

But this squaring step can introduce extra solutions, because there may be values of x for which one side is say $-k$ and the other side is k,s so that when squared, they appear to equal, but $-k \neq k$. So any values found later must be checked in the original equation.

Solving the cubic gives
$$x(x-1)(x+1) = 0,$$

so $x = -1, 0, 1$.

Check these in $\sqrt{x^3 + x^2 - x} = x$. When $x = -1$, we get $1 = -1$, which is false. When $x = 0$, we get $0 = 0$, which is true. When $x = 1$, we get $1 = 1$, which is true.

So the final answers are wrong, and the first error appears in Step (3).

Question 3

In a TMUA question, students are asked to determine which, if any, of the four statements 1, 2, 3 and 4 are true. The options given in the question are:

A: no statements are true

B: only statement 1 is true

C: only statement 2 is true

D: only statements 1 and 2 are true

E: only statement 3 is true

F: only statements 1 and 3 are true

G: only statement 4 is true

H: only statements 2 and 3 are true

I: only statements 2 and 4 are true

Assume that a **competent student** can always correctly determine the truth value of any statement they choose to check.

If the student is lucky, what is the **smallest number of checks** that might let them determine the correct option? Equivalently, what is the fewest checks after which it is **possible** for the student to know the correct option?

A 1

B 2

C 3

D 4

Solution 3

Answer: B

Just checking any one of the statements 1, 2, 3, 4 cannot deduce the answer even if you are lucky. For example, suppose you check statement 1 and it turns out to be true, you would still need to check 2 or 3 to determine which of D or F is the solution.

By the example above, therefore if you check 1 and it is true, then you would only need to make one more check on either 2 or 3 to deduce the answer, so the smallest number of checks possible if you are lukcy is 2, so the answer is B.

Question 4

Find the exact value of

$$\tan 1^\circ \ \tan 2^\circ \ \tan 3^\circ \ \cdots \ \tan 89^\circ.$$

You may find the identity $\sin(x) = \cos(90^\circ - x)$ useful.

Remark: This is a well-known question in mathematical circles, but it is still valuable for students to encounter.

A 1

B 0

C $\sqrt{3}$

D 89

E $\frac{1}{2}$

Solution 4

Answer: A

There has to be some cancellations! Notice that:

$$\tan k^\circ \cdot \tan(90-k)^\circ = \frac{\sin k^\circ}{\cos k^\circ} \cdot \frac{\sin(90-k)^\circ}{\cos(90-k)^\circ} = \frac{\sin k^\circ}{\cos k^\circ} \cdot \frac{\cos k^\circ}{\sin k^\circ} = 1$$

The factors pair up as $(1^\circ, 89^\circ), (2^\circ, 88^\circ), \ldots, (44^\circ, 46^\circ)$, each pair giving 1, and the middle factor is $\tan 45^\circ = 1$. The whole product is therefore 1.

$\tan k^\circ \cdot \tan(90-k)^\circ$ is true in general, $k \neq 90 + 180n$.

Question 5

Let k be a real constant. Consider the polynomial functions

$$p(x) = 2kx^4 - x^2 + 5k \quad \text{and} \quad q(x) = kx^4 + x^2 + k.$$

Which of the following is the **necessary and sufficient** condition on k for $p(x) \geq q(x)$ for all real x?

A $k \geq 0$

B $k > 0$

C $k \geq \frac{1}{4}$

D $k \geq \frac{1}{2}$

E $k > \frac{1}{2}$

F $k \leq -\frac{1}{2}$

G $k \geq \frac{1}{2}$ or $k \leq -\frac{1}{2}$

H $-\frac{1}{2} \leq k \leq \frac{1}{2}$

Solution 5

Answer: D

We need $p(x) \geq q(x)$ for all real x, so

$$2kx^4 - x^2 + 5k \geq kx^4 + x^2 + k.$$

Rearranging gives

$$kx^4 - 2x^2 + 4k \geq 0.$$

This is a quadratic in x^2, let $u = x^2$. Since x is real, $u \geq 0$. So we need

$$ku^2 - 2u + 4k \geq 0$$

for all $u \geq 0$.

If $k \leq 0$, then $ku^2 - 2u + 4k$ cannot be non-negative for all $u \geq 0$. So we need $k > 0$.

For $k > 0$, for the quadratic to be non-negative for all u, we just need $b^2 - 4ac \leq 0$:

$$4 - 16k^2 \leq 0 \quad \Leftrightarrow \quad 4k^2 - 1 \geq 0 \quad \Leftrightarrow \quad (2k-1)(2k+1) \geq 0.$$

Recall $k > 0$, therefore $k \geq \frac{1}{2}$.

Question 6

Given k is a real valued non-zero constant, and x is a non-zero real number, then

$$x^2 = \frac{k}{|x-2|}$$

has n distinct solutions. Which of the following is the **complete** list of possible values of n?

A $0,\ 2,\ 3,\ 4$

B $0,\ 1,\ 3,\ 4$

C $1,\ 2,\ 3,\ 4$

D $0,\ 1,\ 2,\ 3$

E $0,\ 2,\ 3$

F $1,\ 2,\ 3$

G $0,\ 1$

H $1,\ 2$

I $0,\ 3,\ 4$

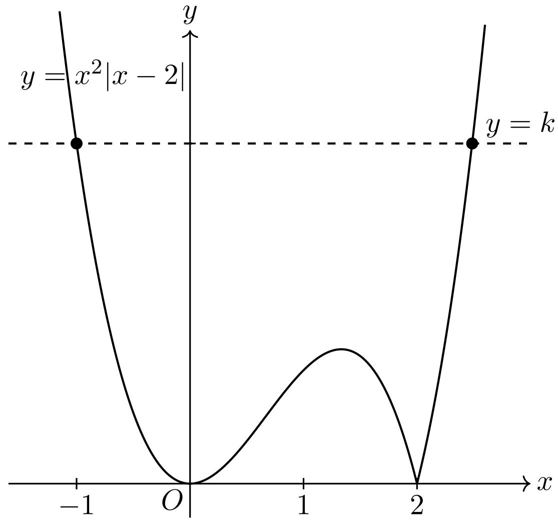
Solution 6

Answer: A

Multiply through by $|x-2| > 0$ (the equation requires $x \neq 2$):

$$x^2|x-2| = k.$$

Now consider the graph of the left and right side of the equation.



To sketch the graph of $y = x^2|x-2|$, sketch $y = x^2(x-2)$ and reflect the $x-2<0$ portion vertically.

By changing the values of k, it is now immediately clear that achievable values of n are 0, 2, 3 and 4.

Question 7

The circle C has equation

$$x^2 + y^2 - 6x + 4y - 12 = 0.$$

The point $P = (11, 2)$ lies outside C. The two tangents from P to C touch the circle at A and B.

Let Γ be the circle passing through P, A and B. What is the radius of Γ?

A $\sqrt{17}$

B $\sqrt{20}$

C $\sqrt{21}$

D $\sqrt{28}$

E $\sqrt{34}$

Solution 7

Answer: B

Complete the square in the equation of C:

$$(x-3)^2 + (y+2)^2 = 25.$$

So the centre of C is $O = (3, -2)$ and its radius is 5.

By a circle theorem, since OPA is a right-angled triangle, the circle C_1 that passes points O, P and A, has OP as a diameter.

Similarly, since OPB is a right-angled triangle, the circle C_2 that passes points O, P and B, also has OP as a diameter.

Therefore C_1 and C_2 are in fact the same circles as they share the diameter OP, and since it passes A, B and P, this circle also must be our Γ, and so

$$OP^2 = (11-3)^2 + (2-(-2))^2 = 8^2 + 4^2 = 80.$$

Therefore the radius of Γ is

$$\frac{OP}{2} = \frac{\sqrt{80}}{2} = \sqrt{20}.$$

Question 8

The following is an attempted proof of the conjecture:

if $\sin\theta > \cos\theta$, **then** $\sin\theta > \frac{1}{\sqrt{2}}$.

Suppose $\sin\theta > \cos\theta$.

Then $\sin\theta - \cos\theta > 0$. (I)

Squaring the positive quantity $\sin\theta - \cos\theta$ gives $(\sin\theta - \cos\theta)^2 > 0$, which expands using $\sin^2\theta + \cos^2\theta = 1$ to $1 - 2\sin\theta\cos\theta > 0$, so $\sin\theta\cos\theta < \frac{1}{2}$. (II)

From (II), $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta < 2$, hence $|\sin\theta + \cos\theta| < \sqrt{2}$, so in particular $\sin\theta + \cos\theta > -\sqrt{2}$. (III)

Adding the inequalities (I) $\sin\theta - \cos\theta > 0$ and (III) $\sin\theta + \cos\theta > -\sqrt{2}$ gives $2\sin\theta > -\sqrt{2}$, so $\sin\theta > -\frac{1}{\sqrt{2}}$; squaring this inequality yields $\sin^2\theta > \frac{1}{2}$. (IV)

Since $\sin^2\theta > \frac{1}{2}$ means $|\sin\theta| > \frac{1}{\sqrt{2}}$, and $\sin\theta > -\frac{1}{\sqrt{2}}$ rules out $\sin\theta < -\frac{1}{\sqrt{2}}$, we conclude $\sin\theta > \frac{1}{\sqrt{2}}$. (V)

Which one of the following is the case?

A The proof is correct.

B The proof is incorrect, and the first error occurs in line (I).

C The proof is incorrect, and the first error occurs in line (II).

D The proof is incorrect, and the first error occurs in line (III).

E The proof is incorrect, and the first error occurs in line (IV).

F The proof is incorrect, and the first error occurs in line (V).

Solution 8

Answer: E

The conjecture is false. Counterexample: $\theta = \frac{5\pi}{6}$ gives $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$, so $\sin\theta > \cos\theta$ holds, but $\sin\theta = \frac{1}{2} < \frac{1}{\sqrt{2}}$.

Lines (I)–(III) are valid.

(I) is just a rearrangement of the hypothesis.

(II) Since $\sin\theta - \cos\theta > 0$, squaring preserves positivity: $(\sin\theta - \cos\theta)^2 > 0$. Expanding and using $\sin^2\theta + \cos^2\theta = 1$ correctly gives $1 - 2\sin\theta\cos\theta > 0$, hence $\sin\theta\cos\theta < \frac{1}{2}$.

(III) $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$, and using (II) this is less than $1 + 1 = 2$, so $|\sin\theta + \cos\theta| < \sqrt{2}$, hence $\sin\theta + \cos\theta > -\sqrt{2}$.

The error is in line (IV). Adding the inequalities is fine and correctly yields $\sin\theta > -\frac{1}{\sqrt{2}}$. The fallacy is the next step: **squaring an inequality does not preserve its direction unless both sides are non-negative**. From $\sin\theta > -\frac{1}{\sqrt{2}}$ one cannot conclude $\sin^2\theta > \frac{1}{2}$; for example $\sin\theta = 0$ satisfies $\sin\theta > -\frac{1}{\sqrt{2}}$ but $\sin^2\theta = 0 < \frac{1}{2}$. Indeed at the counterexample $\theta = \frac{5\pi}{6}$, $\sin\theta = \frac{1}{2} > -\frac{1}{\sqrt{2}}$ but $\sin^2\theta = \frac{1}{4} < \frac{1}{2}$, so (IV) fails.

So (IV) is the first error, and the proof is incorrect.

Question 9

Let k be a real constant. Consider the simultaneous equations

$$k \cdot 2^x + \log_2 y + \log_2 z = 3,$$

$$2^x + k \log_2 y + \log_2 z = 2,$$

$$2^x + \log_2 y + \log_2 z = k,$$

in real unknowns x, y and z, subject to $y > 1$ and $z > 0$.

Which of the following is the **necessary and sufficient** condition on k for the system to have a real-valued solution set $\{x, y, z\}$?

A $k > 1$

B $1 < k < 2$

C $1 < k < 3$

D $2 < k < 3$

E $k < 2$ and $k \neq 1$

F $k > 2$

G $k < 1$ or $k > 2$

H All real k except $k = 1$

Solution 9

Answer: B

Let $X = 2^x$, $Y = \log_2 y$, $Z = \log_2 z$. The constraints translate to $X > 0$, $Y > 0$ (since $y > 1$), and $Z \in \mathbb{R}$ (since $z > 0$). The system becomes

$$kX + Y + Z = 3, \quad X + kY + Z = 2, \quad X + Y + Z = k.$$

Subtract the third equation from the first: $(k-1)X = 3 - k$, so $X = \frac{3-k}{k-1}$ (requires $k \neq 1$).

Subtract the third from the second: $(k-1)Y = 2 - k$, so $Y = \frac{2-k}{k-1}$.

Then $Z = k - X - Y$ is automatically a real number.

Positivity of X:

$$\frac{3-k}{k-1} > 0 \quad \Leftrightarrow \quad 1 < k < 3.$$

Positivity of Y:

$$\frac{2-k}{k-1} > 0 \quad \Leftrightarrow \quad 1 < k < 2.$$

Both must be true, therefore: $1 < k < 2$.

Question 10

The equation

$$\sin x\,(1-\cos^2 x) = k\sin x$$

has exactly n distinct solutions in $[0, 2\pi]$, where k is a real constant.

Which of the following statements are true?

I $\quad 0 < k < 1$ **if** $n = 7$.

II $\quad n = 3$ **only if** $k < 0$ or $k > 1$.

III $\quad n = 5$ and $k \neq 1$ cannot both be true.

A Statement I is the only true statement.

B Statement II is the only true statement.

C Statement III is the only true statement.

D Statements I and III are the only true statements.

E Statements I and II are the only true statements.

F Statements II and III are the only true statements.

G All three statements are true.

H None of them are true.

Solution 10

Answer: D

The equation can be transformed to $\sin x(\sin^2 x - k) = 0$, and be careful not to incorrectly divide out by $\sin x$.

Hence either $\sin x = 0$ or $\sin^2 x = k$. On $[0, 2\pi]$, $\sin x = 0$ gives $x = 0, \pi, 2\pi$, so this always gives 3 solutions. Now a good way to continue is to systematically consider cases of k that lead to different number of solutions.

Now consider $\sin^2 x = k$.

If $k < 0$, there are no solutions from $\sin^2 x = k$, so $n = 3$.

If $k = 0$, the equation $\sin^2 x = 0$ gives the same three solutions already counted, so $n = 3$.

If $0 < k < 1$, the equation $\sin^2 x = k$ gives 4 additional solutions, so $n = 7$.

If $k = 1$, the equation $\sin^2 x = 1$ gives $x = \frac{\pi}{2}, \frac{3\pi}{2}$, so $n = 5$.

If $k > 1$, there are no solutions from $\sin^2 x = k$, so $n = 3$.

Therefore:

$$n = 7 \Leftrightarrow 0 < k < 1,$$

$$n = 5 \Leftrightarrow k = 1,$$

$$n = 3 \Leftrightarrow k \leq 0 \text{ or } k > 1.$$

Statement I is true.

Statement II is false, since $k = 0$ also gives $n = 3$, thus $k < 0$ or $k > 1$ is not wholly necessary.

Statement III is true, since $n = 5$ forces $k = 1$, so $n = 5$ and $k \neq 1$ cannot both be true. This uses the fact that $P \Rightarrow Q$ is equivalent to saying that P and not Q cannot both be true.

So statements I and III are true.

Question 11

Let n be an integer. Which of the following is a **necessary but not sufficient** condition for which $n^3 + 5n$ is divisible by 12?

A n is 1 more than a multiple of 4.

B n is odd.

C n is even.

D n is a multiple of 3.

E n is a multiple of 4.

F n is a multiple of 6.

G n is 2 more than a multiple of 4.

Solution 11

Answer: C

Solution

A good strategy for this question is to start by checking a few small values of n. For $n = 1, 2, 3, 4$, the respective values of $n^3 + 5n$ are 6, 18, 42 and 84. The only multiple of 12 occurs when $n = 4$, so this immediately rules out options A, B, D, F and G.

So the only possible answers are C and E. Also, the case $n = 2$ rules out C being sufficient, therefore the answer must be C.

Remark 1: It can be shown that E is in fact both necessary and sufficient. However, this proof is not needed to deduce the answer to this question.

Remark 2: A **necessary condition** is a consequence of the statement. A very observant student may notice, for example, that condition E implies C. Therefore, if the answer were E, then C would also be a necessary condition, and the answer would not be unique. Hence E can be eliminated as an answer to a **necessary but not sufficient** condition. Similarly, we may immediately eliminate options F and G. This is not strictly needed to solve the question, but it is useful to notice.

Question 12

Let P be a property of a real number x. Each of the following is a statement about all real values of x. **Exactly one** of the following statements is true. Which one is it?

A If $|x| < 1$, then P is true.

B If $|x| < 2$, then P is true.

C If P is true, then $|x| < 1$.

D If P is true, then $|x| < 2$.

E If P is not true, then $x^2 \geq 1$.

F If P is not true, then $x^2 \geq 4$.

G P is true only if $x^2 < 1$.

Solution 12

Answer: D

The fifth statement is the contrapositive of

$$x^2 < 1 \Rightarrow P.$$

Since $x^2 < 1$ is equivalent to $|x| < 1$, the fifth statement is equivalent to the first statement. So neither can be the unique true statement.

The sixth statement is the contrapositive of

$$x^2 < 4 \Rightarrow P.$$

Since $x^2 < 4$ is equivalent to $|x| < 2$, the sixth statement is equivalent to the second statement. So neither can be the unique true statement.

Also, if the second statement is true, then the first statement is true, because $|x| < 1$ implies $|x| < 2$. So the second statement cannot be the unique true statement.

The seventh statement means that if P is true, then $x^2 < 1$. Since $x^2 < 1$ is equivalent to $|x| < 1$, the seventh statement is equivalent to the third statement. So neither can be the unique true statement.

Also, if the third statement is true, then the fourth statement is true, because $|x| < 1$ implies $|x| < 2$. So the third statement cannot be the unique true statement.

Therefore the only statement that can be the unique true statement is the **fourth statement**, which also is not a sufficient condition for any other statements, therefore it confirms to be the answer.

Remark: There is often confusion about a statement such as A. Statement A means: whenever x is between -1 and 1, P must be true. Statement B makes the stronger claim that whenever x is between -2 and 2, P must be true. Therefore B implies A, because every value satisfying $|x| < 1$ also satisfies $|x| < 2$. However, A does not imply B. For example, A could be true while P is false when $x = 1.5$. This would not contradict A, since 1.5 is not in the range $|x| < 1$, but it would contradict B, since 1.5 is in the range $|x| < 2$.

Question 13

Positive real numbers x, y, z, none equal to 1, satisfy

$$x^{\log_y z} = y^{\log_z x} = z^{\log_x y} = 8.$$

What is the value of $(\log_2 x)(\log_2 y)(\log_2 z)$?

A 27

B 18

C 8

D 16

E 32

F not possible to determine

Solution 13

Answer: A

Let $a = \log_2 x$, $b = \log_2 y$, and $c = \log_2 z$. Then $x = 2^a$, $y = 2^b$, and $z = 2^c$.

Since x, y, and z are not equal to 1, we have $a, b, c \neq 0$.

Now

$$y^{\frac{c}{b}} = (2^b)^{\frac{c}{b}} = 2^c = z.$$

So $\log_y z = \frac{c}{b}$. Hence

$$x^{\log_y z} = (2^a)^{\frac{c}{b}} = 2^{\frac{ac}{b}}.$$

Since $x^{\log_y z} = 8 = 2^3$, we get

$$\frac{ac}{b} = 3.$$

Similarly, due to symmetry we get $\frac{ab}{c} = 3$ and $\frac{bc}{a} = 3$.

Multiplying the three equations gives

$$\frac{ac}{b} \cdot \frac{ab}{c} \cdot \frac{bc}{a} = 27 \quad \Leftrightarrow \quad abc = 27 \quad \Leftrightarrow (\log_2 x)(\log_2 y)(\log_2 z) = 27.$$

Question 14

For each positive integer k, let $F(k)$ be the sum of all real solutions x of

$$\sin(2^k x) = \frac{1}{2}$$

in the interval $0 \leq x \leq \frac{2\pi}{2^k}$. Find

$$\sum_{k=1}^{\infty} F(k).$$

A $\frac{\pi}{2}$

B $\frac{3\pi}{2}$

C π

D $\frac{3\pi}{4}$

E $\frac{5\pi}{2}$

F 2π

G The series diverges.

Solution 14

Answer: C

For $k = 1$, there are exactly two solutions:

$$\frac{\pi}{6} \times \frac{1}{2} \text{ and } \frac{5\pi}{6} \times \frac{1}{2}.$$

For $k = 2$, because of the domain restriction, there are also exactly two solutions:

$$\frac{\pi}{6} \times \frac{1}{2^2} \text{ and } \frac{5\pi}{6} \times \frac{1}{2^2}.$$

Similarly for $k = 3$, because of the domain restriction, there are also exactly two solutions:

$$\frac{\pi}{6} \times \frac{1}{2^3} \text{ and } \frac{5\pi}{6} \times \frac{1}{2^3}.$$

The same pattern continues.

Therefore: $F(1) = \pi \times \frac{1}{2}$, $F(2) = \pi \times \frac{1}{2^2}$ and $F(3) = \pi \times \frac{1}{2^3}$. They form a geometric sequence with first term $\frac{\pi}{2}$, common ratio $1/2$. Thus

$$\sum_{k=1}^{\infty} F(k) = \frac{\frac{\pi}{2}}{1 - \frac{1}{2}} = \pi.$$

Question 15

A set X of positive integers, with $|X| \geq 3$, is called **cross-linked** if and only if for every number x in X there exist two distinct numbers y and z in X, both different from x, and a prime p, such that p divides each of x, y and z.

Let Y be a set of positive integers with $|Y| \geq 3$.

Which of the following statements is equivalent to Y being **not** cross-linked?

A For every number x in Y, and for every two distinct numbers y and z in Y with $y \neq x$ and $z \neq x$, there is no prime which divides all of x, y and z.

B For every number x in Y there exist two distinct numbers y and z in Y, both different from x, such that no prime divides all of x, y and z.

C There exists a number x in Y, and there exist two distinct numbers y and z in Y with $y \neq x$ and $z \neq x$, such that no prime divides all of x, y and z.

D There exists a number x in Y such that for every two distinct numbers y and z in Y with $y \neq x$ and $z \neq x$, there is no prime which divides all of x, y and z.

E There exists a number x in Y such that for every two distinct numbers y and z in Y with $y \neq x$ and $z \neq x$, there is a prime which divides x but divides neither y nor z.

F For every number x in Y there exist two distinct numbers y and z in Y, both different from x, and a prime p, such that p divides x but p does not divide y and p does not divide z.

Solution 15

Answer: D

A good way to approach this question is to simply write down the negation of cross-linked.

Start outwards. Y is cross-linked requires some property p_1 to be true for every x in Y. Therefore the negation is that there exists an x in Y for which this property p_1 is false.

The property p_1: there exist distinct y and z in Y, different from x, for which some property p_2 is true. Therefore not having the property p_1, or the negation of p_1, is: there does not exist such y and z with property p_2. Another way of writing the same idea: if you examine every pair of possible y and z in Y, then none of them have property p_2.

The property p_2: there exists a prime p that divides x, y and z. Not having property p_2, or the negation of p_2: for every prime p, p does not divide all of x, y and z, or equivalently, no prime p divides all of x, y and z.

Altogether, a valid negation is:

There exists a number x in Y such that for every two distinct numbers y and z in Y with $y \neq x$ and $z \neq x$, there is no prime which divides all of x, y and z.

This is option **D**.

Question 16

Let

$$W = \int_0^1 4^{\sqrt{x}}\,dx, \qquad X = \int_1^2 2^{\sqrt{x-1}}\,dx, \qquad Y = \int_0^{1/2} 2\cdot 2^{2x}\,dx, \qquad Z = \int_{-1}^1 \frac{1}{2}(\sqrt{2})^{(x+1)/2}\,dx.$$

Which of the following gives W, X, Y, Z in order from **smallest to largest**?

A $Z < Y < X < W$

B $W < X < Y < Z$

C $Z < X < Y < W$

D $Y < Z < X < W$

E $Z < Y < W < X$

F $X < Z < Y < W$

Solution 16

Answer: A

This question is about using transformations to transform the integrals to a comparable form.

The integral $W = \int_0^1 4^{\sqrt{x}}\,dx = \int_0^1 2^{2\sqrt{x}}\,dx$.

The integral $X = \int_1^2 2^{\sqrt{x-1}}\,dx$, we transform to the left by 1 unit, this changes x to $x+1$, and move the range $[1,2]$ to $[0,1]$. So that $X = \int_0^1 2^{\sqrt{x}}\,dx$.

The integral $Y = \int_0^{1/2} 2\cdot 2^{2x}\,dx$, we transform by stretching it horizontally by factor of 2, to enlarge the range from $[0,1/2]$ to $[0,1]$, this changes x to $x/2$, to make the area the same as before, we also have to multiply by $1/2$. Therefore

$$Y = \int_0^{1/2} 2\cdot 2^{2x}\,dx = \int_0^1 2\cdot 2^{2(x/2)}\,dx \times \frac{1}{2} = \int_0^1 2^x\,dx.$$

The integral $Z = \int_{-1}^1 \frac{1}{2}(\sqrt{2})^{(x+1)/2}\,dx$ is the most tricky, but its range tell us a sensible transformation scheme: $[-1,1]$ to $[0,2]$, then $[0,2]$ to $[0,1]$. The first transformation changes x to $x-1$, the second changes x to $2x$ which is a horizontal stretch by scale factor $1/2$, thus we also need to times the area by 2 to preserve the original area. Therefore

$$Z = \int_0^2 \frac{1}{2}(\sqrt{2})^{(x-1+1)/2}\,dx = \int_0^1 \frac{1}{2}(\sqrt{2})^{(2x)/2}\,dx \times 2 = \int_0^1 (\sqrt{2})^x\,dx = \int_0^1 2^{x/2}\,dx.$$

So the question is equivalent to comparing

$$W = \int_0^1 2^{2\sqrt{x}}\,dx, \quad X = \int_0^1 2^{\sqrt{x}}\,dx, \quad Y = \int_0^1 2^x\,dx, \quad Z = \int_0^1 2^{x/2}\,dx.$$

Since 2^t is strictly increasing, the ordering of the four integrands on $(0, 1)$ is determined by the ordering of their exponents:

$$2\sqrt{x} \geq \sqrt{x} \geq x \geq \frac{x}{2},$$

with equality only sometimes occurring on the end points 0 and 1, this is enough for us to immediately deduce:

$$W > X > Y > Z \quad \text{or} \quad Z < Y < X < W.$$

Remark: You can derive at the comparable integrals also by **integration by substitution**, though the intention here is to get the student to do the question using transformations, this is faster if you are familiar with the technique.

Question 17

Find the units digit of

$$7^{7^{7^{7^7}}}.$$

You may find it useful to consider the remainder of odd powers of 7 when divided by 4.

A 1

B 2

C 3

D 4

E 5

F 6

G 0

Solution 17

Answer: C

The units digits of powers of 7 repeat in a cycle:

$$7^1 \text{ ends in } 7, \quad 7^2 \text{ ends in } 9, \quad 7^3 \text{ ends in } 3, \quad 7^4 \text{ ends in } 1.$$

So we only need to know where the exponent

$$7^{7^{7^7}}$$

falls in this cycle of length 4.

The key to the question is that: **any odd power of 7 is one less than a multiple of 4**.

Since $7 = 8 - 1$, we can write an odd power of 7 as

$$7^{2k+1} = (8-1)^{2k+1}.$$

When this is expanded, every term except the final term contains a factor of 8, so all those terms together form a multiple of 8, and hence a multiple of 4.

The final term is

$$(-1)^{2k+1} = -1.$$

Therefore

$$(8-1)^{2k+1} = \text{a multiple of } 4 - 1.$$

So any odd power of 7 is one less than a multiple of 4.

Since apply the result to 7^7, so it is one less than a multiple of 4, so it is odd, then 7^{7^7} is also one less than a multiple of 4, repeatedly apply this, then the overall exponent is also 1 less than a multiple of 4, therefore it is in the third position of the cycle. So the units digit is **3**.

This result can be applied recursively! The number 7^7 is an odd power of 7, so it is one less than a multiple of 4. In particular, it is odd.

Therefore 7^{7^7} is also an odd power of 7, so it is also one less than a multiple of 4. Repeating the same argument, it does not matter whether the tower contains 3 copies of 7 or 10 copies of 7: each new power is still an odd power of 7, and is therefore one less than a multiple of 4.

Hence $7^{7^{7^{7^{7}}}}$ is one less than a multiple of 4, so it is in the third position of the units digit cycle. Therefore the units digit is 3.

Question 18

Let f and g be polynomials. Consider the following four statements.

I: If $f'(x) \geq g'(x)$ for all $x \geq 0$ and $f(0) \geq g(0)$, then $f(x) \geq g(x)$ for all $x \geq 0$.

II: If $f(x) \geq g(x)$ for all $x \geq 0$, then $\int_0^x t\, f(t)\, \mathrm{d}t \geq \int_0^x t\, g(t)\, \mathrm{d}t$ for all $x \geq 0$.

III: If $\int_0^x f(t)\, \mathrm{d}t \geq \int_0^x g(t)\, \mathrm{d}t$ for all $x \geq 0$, then $f(x) \geq g(x)$ for all $x \geq 0$.

IV: If $\int_0^x f(t)\, \mathrm{d}t \geq \int_0^x g(t)\, \mathrm{d}t$ for all $x \geq 0$, then $\int_0^x t^2\, f(t)\, \mathrm{d}t \geq \int_0^x t^2\, g(t)\, \mathrm{d}t$ for all $x \geq 0$.

Which of the statements are true?

A None of them

B I only

C II only

D III only

E IV only

F I and II only

G II and III only

H I and III only

I all except I

J all except II

K all except III

L all except IV

Solution 18

Answer: F

This is a far harder version of a question from the past papers!

Let $h(x) = f(x) - g(x)$. The four statements can then be considered in terms of h.

I is TRUE. If $f'(x) \geq g'(x)$ for all $x \geq 0$, then $h'(x) \geq 0$ for all $x \geq 0$. Also, $f(0) \geq g(0)$ means $h(0) \geq 0$. Since $h'(x) \geq 0$ on $x \geq 0$, h is increasing on this interval. Therefore for all $x \geq 0$, we have $h(x) \geq h(0) \geq 0$, so $f(x) \geq g(x)$.

II is TRUE. If $f(x) \geq g(x)$ for all $x \geq 0$, then $h(x) \geq 0$ for all $x \geq 0$. Also, for $0 \leq t \leq x$, we have $t \geq 0$. Hence $t \cdot h(t) \geq 0$, so $\int_0^x t \cdot h(t)\,\mathrm{d}t \geq 0$. Therefore $\int_0^x t \cdot f(t)\,\mathrm{d}t \geq \int_0^x t \cdot g(t)\,\mathrm{d}t$.

III is FALSE. This is harder to see, as it looks rather plausible at first sight. The condition says that the accumulated area of $f - g$ from 0 to x is always non-negative. This does not mean that $f(x) - g(x)$ itself is always non-negative.

For counterexample, take $g(x) = 0$ and $f(x) = (x-1)(x-2)$. Here the area between 0 and 1 is positive, and is visibly greater than the area between 1 and 2 which is below the x-axis. for $x > 2$, $f(x)$ is positive. Therefore it is not hard to see that $\int_0^x h(t)\,\mathrm{d}t \geq 0$ for all $x \geq 0$. However, h is negative between 1 and 2.

IV is FALSE. This is even harder to see, because it looks like a weighted version of III by a positive quantity t^2, so it may look true at first, but it's false in general! The key point that makes this sometimes false, is that the factor t^2 is a non-constant weighting factor. It gives **less** weight to values of t between 0 and 1, and it gives **more** weight to values of $t > 1$ to our $f(t) - g(t)$. This can so it can change the balance between the positive and negative parts.

The same counterexample works. Take $g(x) = 0$ and $f(x) = (x-1)(x-2)$. Then it can be checked that

$$\int_0^1 t^2(t-1)(t-2)\ \mathrm{d}t = \frac{7}{60} \quad \text{and} \quad \int_1^2 t^2(t-1)(t-2)\ \mathrm{d}t = -\frac{23}{60},$$

therefore for there is some x in interval $(1, 2)$, for which $\int_0^x t^2(t-1)(t-2)\ \mathrm{d}t < 0$.

You do not necessarily need to construct an actual counterexample, it is enough to deduce counterexamples should exist from the effect of t^2 as weighting factor.

In the end, the true statements are I and II only.

Question 19

A circle has centre $A = (6, 7)$. Two tangents from the point $B = (1, 2)$ touch the circle at P and Q. Given that length $BP = 2\sqrt{10}$, find the sum of the gradients of the tangent lines BP and BQ.

A $\frac{6}{3}$

B $\frac{3}{4}$

C $\frac{8}{3}$

D $\frac{3}{2}$

E $\frac{10}{3}$

F $\frac{7}{4}$

G $\frac{5}{4}$

Solution 19

Answer: E

Translate everything by the vector $(-1, -2)$. This does not change any gradients, so we can work in the transformed universe which has much nicer numbers!

The point $B = (1, 2)$ becomes $O = (0, 0)$, and the centre $A = (6, 7)$ becomes $C = (5, 5)$.

Let the radius of the circle be r. Since $OP = 2\sqrt{10}$ and $OC = \sqrt{5^2 + 5^2} = 5\sqrt{2}$, the right triangle OCP gives

$$OC^2 = OP^2 + r^2.$$

So $50 = 40 + r^2$, hence $r^2 = 10$. The transformed circle is

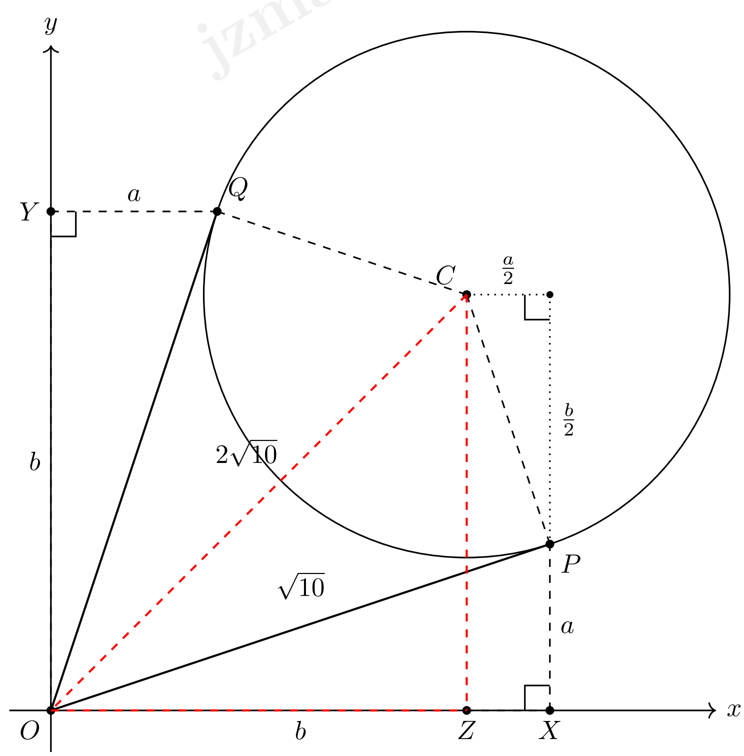
$$(x - 5)^2 + (y - 5)^2 = 10.$$

A tangent line through the origin has equation $y = mx$. Substitute the tangent equation into the circle equation:

$$(x - 5)^2 + (mx - 5)^2 = 10.$$

The expanded quadratic is
$$(m^2+1)x^2 - 10(m+1)x + 40 = 0.$$
For the line to be a tangent, this quadratic must have exactly one solution for x, so its discriminant must be 0:
$$[-10(m+1)]^2 - 4(m^2+1)(40) = 0.$$
So
$$100(m+1)^2 - 160(m^2+1) = 0.$$
Dividing by 20 gives
$$5(m+1)^2 - 8(m^2+1) = 0.$$
Expanding,
$$5m^2 + 10m + 5 - 8m^2 - 8 = 0,$$
so
$$3m^2 - 10m + 3 = 0.$$
The two tangent lines have gradients equal to the two roots of this quadratic. Therefore the sum of the gradients is $\frac{10}{3}$.

Altenative geometrical method



It is possible to find the answer to this geometrically. Work with the transformed version of the problem, centre $(5,5)$, two tangents both passing O. Let P be the lower of the two points, drop a perpendicular vertical line from P to the x-axis, let X be the point where it cross the x-axis. Let $PX = a$ and $OX = b$. By symmetry, we can draw a horizontal line from Q to the y-axis, call the point of intersection Y, and then $QY = a$ and $OY = b$. The sum of gradients is then just

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} = \frac{(2\sqrt{10})^2}{ab}.$$

Therefore, it is enough to find ab.

Next observe that crucially, if we construct a right angled triangle with XC being the hypotenuse, by drawning two lines, one horizontal one vertical, then this triangle is **similar** to OXP, by scale factor $1/2$, which we get by comparing the lengths of their hypotenuses. Thus its other sides are $a/2$ and $b/2$. Next bring these together into the isosceles right-angled triangle with hypotenuse OC, so that we deduce:

$$b/2 + a = 5 \text{ and } b - a/2 = 5.$$

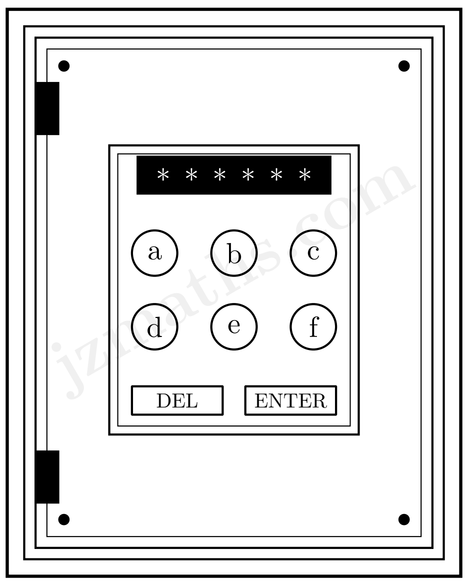
From which we find $b = 6$ and $a = 2$, hence

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} = \frac{(2\sqrt{10})^2}{12} = \frac{10}{3}.$$

Question 20

Disclaimer: This is a **very hard** version of a similar question from the past exam questions! Expect to give yourself an extra 10-15 minutes to solve this. You will learn a great deal from trying!

A safe is protected by a 6-character password that uses each of the letters a, b, c, d, e, f exactly once. After each attempt the safe reports only the number of positions in which the attempted string agrees with the password (it does not say which positions). The following three attempts have already been made: $abcdef$ gave 0 correct positions, $feadbc$ gave 0 correct positions, and $fbdcae$ gave 3 correct positions. A player now plays optimally, choosing each subsequent attempt with full knowledge of all responses received so far. What is the smallest number N such that the player is guaranteed to be able to identify the password after at most N further attempts (regardless of what the password actually is)?



A 1

B 2

C 3

D 4

E more than 5

Solution 20

Answer: B

Throughout, write a candidate password as $\pi_1\pi_2\pi_3\pi_4\pi_5\pi_6$, a permutation of $\{a, b, c, d, e, f\}$.

Step 1: Position constraints from the first two attempts.

Each attempt giving 0 correct positions tells us, for every position i, that the letter placed there is not π_i. The attempt $abcdef$ rules out $\pi_1 = a$, $\pi_2 = b$, $\pi_3 = c$, $\pi_4 = d$, $\pi_5 = e$, $\pi_6 = f$. The attempt $feadbc$ rules out $\pi_1 = f$, $\pi_2 = e$, $\pi_3 = a$, $\pi_4 = d$, $\pi_5 = b$, $\pi_6 = c$. Combining,

$$\pi_1 \notin \{a, f\}, \quad \pi_2 \notin \{b, e\}, \quad \pi_3 \notin \{a, c\}, \quad \pi_4 \neq d, \quad \pi_5 \notin \{b, e\}, \quad \pi_6 \notin \{c, f\}.$$

Step 2: Combine with the third attempt.

The attempt $fbdcae$ has 3 correct positions, so exactly 3 of the six claims

$$\pi_1 = f, \quad \pi_2 = b, \quad \pi_3 = d, \quad \pi_4 = c, \quad \pi_5 = a, \quad \pi_6 = e$$

hold. But $\pi_1 = f$ and $\pi_2 = b$ are already ruled out by Step 1, so both must be false. Exactly 3 of the remaining 4 claims must therefore hold; equivalently, exactly one of them fails.

Step 3: Enumerate the candidates.

For each choice of "false claim", the other three letters are pinned and we fill in the remaining three positions using the unused letters, subject to the Step 1 constraints.

Case A, $\pi_6 \neq e$: pin $\pi_3 = d$, $\pi_4 = c$, $\pi_5 = a$. Letters $\{b, e, f\}$ remain for positions $\{1, 2, 6\}$. The constraint $\pi_2 \notin \{b, e\}$ forces $\pi_2 = f$; then $\pi_6 \notin \{c, e, f\}$ forces $\pi_6 = b$, leaving $\pi_1 = e$. Candidate: $efdcab$.

Case B, $\pi_5 \neq a$: pin $\pi_3 = d$, $\pi_4 = c$, $\pi_6 = e$. Letters $\{a, b, f\}$ remain. Then $\pi_1 \notin \{a, f\}$ forces $\pi_1 = b$; $\pi_5 \notin \{a, b, e\}$ forces $\pi_5 = f$, leaving $\pi_2 = a$. Candidate: $badcfe$.

Case C, $\pi_4 \neq c$: pin $\pi_3 = d$, $\pi_5 = a$, $\pi_6 = e$. Letters $\{b, c, f\}$ remain for $\{1, 2, 4\}$ with $\pi_1 \in \{b, c\}$, $\pi_2 \in \{c, f\}$, $\pi_4 \in \{b, f\}$. Two valid assignments: $bcdfae$ and $cfdbae$.

Case D, $\pi_3 \neq d$: pin $\pi_4 = c$, $\pi_5 = a$, $\pi_6 = e$. Letters $\{b, d, f\}$ remain for $\{1, 2, 3\}$ with $\pi_1 \in \{b, d\}$, $\pi_2 \in \{d, f\}$, $\pi_3 \in \{b, f\}$. Two valid assignments: $bdfcae$ and $dfbcae$.

So the candidate set has six elements:

$$\{badcfe,\ bcdfae,\ bdfcae,\ cfdbae,\ dfbcae,\ efdcab\}.$$

Step 4: One further attempt cannot guarantee identification.

First observe that any two of the six remaining candidates agree in at least two positions. Now consider any one further attempt. If two candidates agree in a certain position, then that position contributes the same amount to both of their response scores: either the attempt matches both of them there, or it matches neither of them there. Since any two candidates agree in at least 2 positions, they can differ in at most 4 positions. Therefore, under any single attempt, the response scores of any two candidates can differ by at most 4.

So the range of the six possible response scores is at most 4, implies at least two of them will have to respond with the same response, and therefore it is not possible to determine the password with a single further attempt, and hence $N \geq 2$.

Step 5: Two further attempts suffice.

We now show that two further attempts are enough.

The high-level idea is to choose a test attempt that splits the six candidates into useful response groups. A good test should not be too similar to many candidates, otherwise many candidates may give the same high response. For example, a test with $bfdcae$ that gives five candidates the same response of 4 would be poor. But it should also not be too far away from all candidates, because if all candidates returned 0, we would learn nothing as well.

So the aim is to choose a test whose responses divide the candidates into manageable buckets. There are many possible attempts that can work; we only need to exhibit one valid strategy.

For attempt 4, we could for example try $abcfde$. The six candidates give responses

$$badcfe \to 1, \quad bcdfae \to 2, \quad bdfcae \to 1, \quad cfdbae \to 1, \quad dfbcae \to 1, \quad efdcab \to 0.$$

A response of 0 identifies the password as $efdcab$; a response of 2 identifies it as $bcdfae$. If the response is 1, the candidate set narrows to $\{badcfe, bdfcae, cfdbae, dfbcae\}$.

In that case, for attempt 5, try $baecdf$. The four remaining candidates give

$$badcfe \to 3, \quad bdfcae \to 2, \quad cfdbae \to 0, \quad dfbcae \to 1,$$

all distinct, so the password is identified.

Hence two further attempts always suffice, and combining with Step 4, $N = 2$. The answer is **B**. I hope you enjoyed that!