JZ Mock Set C Paper 1 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

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Question 1

Find

$$\int_{-1}^{1} 1 + 2x + 3x^2 + 4x^3 + ... + 11x^{10}\,dx.$$

A 11

B 10

C 12

D 22

E 0

F 2

Solution 1

Answer: C

The odd powered terms such as $4x^3$ integrate to 0 over $[-1, 1]$ since they are odd functions.

So we only need to consider the even powers:

$$\int_{-1}^{1} 1 + 3x^2 + 5x^4 + 7x^6 + 9x^8 + 11x^{10}\,dx.$$

For each even power term $(r + 1)x^r$, where $r = 0, 2, 4, 6, 8, 10$, we have

$$\int_{-1}^{1} (r+1)x^r\,dx = 2\int_{0}^{1} (r+1)x^r\,dx = 2[x^{r+1}]_0^1 = 2.$$

There are 6 even power terms, so the integral is $6 \cdot 2 = 12$.

Question 2

Find the number of real solutions of $|x^2 - 1| = 2^x$.

A 1

B 2

C 3

D 4

E 5

F 6

Solution 2

Answer: E

Sketch the graphs of $y = |x^2 - 1|$ and $y = 2^x$.

From the sketch, we can see that $x = 0$ is a solution. There are also two solutions with $x < 0$, and **2** solutions with $x > 0$.

On the right of 0, there are two solutions because the graph of $y = x^2 - 1$ initially catches up with $y = 2^x$, but eventually 2^x grows faster. This can be observed from the graph, so we do not need to find the derivatives explicitly.

Therefore, the total number of real solutions is $1 + 2 + 2 = 5$.

Question 3

The real numbers a, b, c satisfy

$$4 - |a-8| \geq 0, \qquad |2b-10| \leq 6, \qquad |c-8| \leq 7.$$

What is the least possible value of $|abc|$?

A 15

B 12

C 10

D 32

E 8

F 0

Solution 3

Answer: E

Solve each:

$$4 - |a-8| \geq 0 \iff |a-8| \leq 4 \iff 4 \leq a \leq 12,$$

$$|2b-10| \leq 6 \iff -6 \leq 2b-10 \leq 6 \iff 2 \leq b \leq 8,$$

$$|c-8| \leq 7 \iff 1 \leq c \leq 15.$$

Since $|abc| = |a|\,|b|\,|c|$, $|abc|$ is minimised by minimising each factor separately. Hence the least possible value of $|abc|$ is $4 \times 2 \times 1 = 8$.

Question 4

Which of the following values is the **largest**?

A $\frac{7}{5}$

B $\frac{17}{12}$

C $\frac{10}{7}$

D $\frac{11}{8}$

E $2\cos\left(\frac{\pi}{4}\right)$

Solution 4

Answer: C

First note that

$$2\cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}.$$

Next, we compare fractions by cross multiply, compare against $\frac{10}{7}$ as it has a 10 for easy multiplication.

We get $\frac{10}{7} > \frac{7}{5}$ because $10 \cdot 5 > 7 \cdot 7$.

Also $\frac{10}{7} > \frac{17}{12}$ because $10 \cdot 12 > 17 \cdot 7$.

Similarly $\frac{10}{7} > \frac{11}{8}$ because $10 \cdot 8 > 11 \cdot 7$.

Finally, since numbers are positive, we can compare squares: $\left(\frac{10}{7}\right)^2 = \frac{100}{49} > 2$. Therefore $\frac{10}{7} > \sqrt{2}$ and so the largest value is $\frac{10}{7}$.

Question 5

How many distinct real solutions does $4^x + 6^x = 9^x$ have?

A 0

B 1

C 2

D 3

E 4

Solution 5

Answer: B

Compare the graphs of $y = 4^x + 6^x$ and $y = 9^x$. There is a solution between 1 and 2, since

$$4^1 + 6^1 > 9^1$$

but

$$4^2 + 6^2 < 9^2.$$

After this crossing, 9^x grows faster than both 4^x and 6^x, so once 9^x has overtaken $4^x + 6^x$, it cannot be caught again. This suggests that there is exactly one real solution.

Remark: This question belongs to a well-known class of problems. The usual version is more demanding and asks for the value of x. This is how it is done.

Divide through by 9^x:

$$\left(\frac{4}{9}\right)^x + \left(\frac{6}{9}\right)^x = 1, \qquad \text{that is} \qquad \left(\frac{2}{3}\right)^{2x} + \left(\frac{2}{3}\right)^x = 1.$$

Let $u = \left(\frac{2}{3}\right)^x$. Then $u > 0$ and

$$u^2 + u - 1 = 0,$$

whose only positive root is

$$u = \frac{\sqrt{5} - 1}{2}.$$

Since $\left(\frac{2}{3}\right)^x$ takes every positive value exactly once, there is exactly one real solution.

Question 6

Three distinct numbers are chosen at random from $\{1, 2, \ldots, 10\}$. Find the probability that, arranged in increasing order, they form an arithmetic progression.

A $\frac{1}{4}$

B $\frac{1}{5}$

C $\frac{1}{6}$

D $\frac{1}{8}$

E $\frac{2}{9}$

F $\frac{2}{15}$

Solution 6

Answer: C

There are $\binom{10}{3} = 120$ equally likely choices.

An increasing progression $a, a + d, a + 2d$ needs $d \geq 1$ and $a + 2d \leq 10$.

Counting by common difference, $d = 1$ gives 8 progressions, $d = 2$ gives 6, $d = 3$ gives 4 and $d = 4$ gives 2, a total of 20. The probability is $\frac{20}{120} = \frac{1}{6}$.

Question 7

A line l with positive gradient is a common tangent to the circles $x^2 + y^2 = 4$ and $(x-9)^2 + y^2 = 25$, such that the two circles lie on the same side of l.

Find the gradient of l.

A $\frac{1}{3}$

B $2\sqrt{2}$

C $\frac{\sqrt{2}}{4}$

D $\frac{3\sqrt{2}}{8}$

E $\frac{\sqrt{2}}{2}$

F $\frac{2\sqrt{2}}{3}$

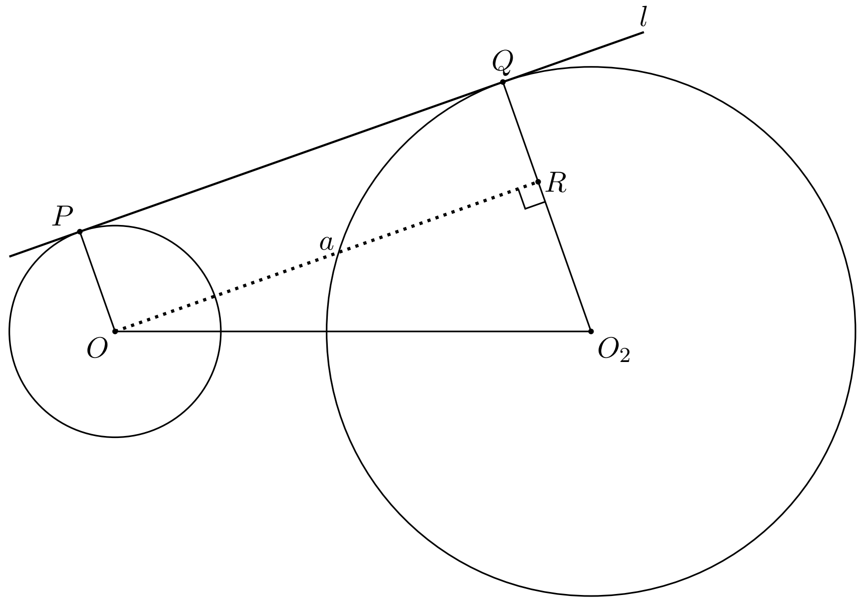
G $\frac{3\sqrt{2}}{4}$

H $\frac{7\sqrt{2}}{8}$

Solution 7

Answer: C

Let $O = (0,0)$ be the centre of $x^2 + y^2 = 4$, and let $O_2 = (9,0)$ be the centre of $(x-9)^2 + y^2 = 25$.



Since l is tangent to both circles, OP and O_2Q are perpendicular to l. Their lengths are the radii of the circles, so $OP = 2$ and $O_2Q = 5$.

Draw OR parallel to l, and draw O_2R perpendicular to OR, as in the diagram. Then O_2R represents the difference between the perpendicular distances from the two centres to the tangent line, so

$$O_2R = 5 - 2 = 3.$$

Let $OR = a$. Since $OR \perp O_2R$, triangle OO_2R is right-angled at R. Also $OO_2 = 9$, because the centres are $(0,0)$ and $(9,0)$.

By Pythagoras,

$$a^2 + 3^2 = 9^2.$$

So

$$a^2 = 72,$$

hence

$$a = 6\sqrt{2}.$$

Now let θ be the angle that l makes with the positive x-axis. Since OR is parallel to l, the gradient of l is $\tan\theta$. In triangle OO_2R,

$$\tan\theta = \frac{O_2R}{OR} = \frac{3}{6\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Therefore the gradient of l is $\frac{\sqrt{2}}{4}$.

Remark: If you are not aware of the fact that the gradient of l is $\tan\theta$, you can use a different approach. Drop a perpendicular from R down to OO_2, call the point of intersection X, then ORX and ORO_2 are similar triangles. The required gradient is

$$\frac{XR}{OX} = \frac{O_2R}{OR}.$$

Question 8

For positive integers n let:

$$I(n) = \int_0^{2n} \frac{|n-x|}{n} \, dx$$

What is the largest positive integer n for which

$$\sum_{r=1}^{n} I(r) < 210?$$

A 15

B 16

C 19

D 20

E 21

Solution 8

Answer: C

For any positive integer k,

$$I(k) = \int_0^{2k} \frac{|k-x|}{k} \, dx.$$

The graph of $y = |k-x|$ from $x = 0$ to $x = 2k$ forms two right-angled triangles, each with base k and height k. Therefore

$$\int_0^{2k} |k-x| \, dx = 2 \cdot \frac{1}{2} k^2 = k^2.$$

So

$$I(k) = \frac{k^2}{k} = k.$$

Hence

$$\sum_{r=1}^{n} I(r) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Therefore we need $\frac{n(n+1)}{2} < 210$ or $n(n+1) < 420$.

Trial and error: $19 \cdot 20 = 380 < 420$, $20 \cdot 21 = 420$.

So $n = 20$ gives equality, not a value less than 210. Therefore the largest possible positive integer is 19.

Question 9

Three players A, B and C take turns throwing a fair six-sided die in the order A, B, C, A, B, C, ... until one of them wins. The win conditions on a player's own throw are:

$$\begin{aligned}&A \text{ wins if } A \text{ throws a } 6,\\&B \text{ wins if } B \text{ throws a } 5 \text{ or a } 6,\\&C \text{ wins if } C \text{ throws a } 4, 5 \text{ or } 6.\end{aligned}$$

If a player throws and does not win, play passes to the next player. What is the probability that B is the first to win?

A $\frac{3}{13}$

B $\frac{5}{18}$

C $\frac{5}{13}$

D $\frac{6}{13}$

E $\frac{1}{3}$

F $\frac{5}{9}$

Solution 9

Answer: C

Consider one full round (the three throws by A, then B, then C).

The probability that B wins on the first round is the probability that A fails, then B succeeds:

$$P(B \text{ wins in round } 1) = \frac{5}{6} \cdot \frac{2}{6} = \frac{5}{18}.$$

The probability that no one wins in a round is

$$P(\text{no winner in a round}) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{60}{216} = \frac{5}{18}.$$

If no one wins in a round, the situation resets identically. So

$$P(B \text{ wins}) = P(B \text{ win in round } 1) + P(B \text{ win in round } 2) + P(B \text{ win in round } 3) + \cdots,$$

therefore

$$P(B \text{ wins}) = \frac{5}{18} + \left(\frac{5}{18}\right)^1 \cdot \frac{5}{18} + \left(\frac{5}{18}\right)^2 \cdot \frac{5}{18} + \cdots$$

This is a geometric series with first term $\frac{5}{18}$ and common ratio $\frac{5}{18}$:

$$P(B \text{ wins}) = \frac{\frac{5}{18}}{1 - \frac{5}{18}} = \frac{\frac{5}{18}}{\frac{13}{18}} = \frac{5}{13}.$$

Question 10

The curve $x = (y-2)^2 - 1$ is transformed as follows:

First, it is translated vertically in the positive y-direction by 2 units.

Next, it is reflected in the line $x = 2$.

Finally, it is rotated 90° clockwise about the point $(1, 1)$.

Which of the following is the equation of the resulting curve?

A $y = x^2 - 8x + 13$

B $y = x^2 - 8x + 19$

C $y = -x^2 + 8x - 13$

D $x = y^2 - 8y + 15$

E $y = x^2 - 4x + 1$

Solution 10

Answer: A

Our strategy in this question is to keep track of the apex and direction of the quadratic curve through the transformations.

The original curve $x = (y-2)^2 - 1$ is a sideways quadratic. Its apex is $(-1, 2)$, and it faces right, since x is equal to a positive square expression.

After translating vertically in the positive y-direction by 2 units, the apex moves from $(-1, 2)$ to $(-1, 4)$. The curve still faces right.

Next, the curve is reflected in the line $x = 2$. The apex $(-1, 4)$ is reflected to $(5, 4)$, because -1 is 3 units to the left of $x = 2$, so its reflection is 3 units to the right of $x = 2$. The curve now faces left.

Finally, the curve is rotated 90° clockwise about $(1, 1)$. Relative to $(1, 1)$, the apex $(5, 4)$ has displacement $(4, 3)$. A 90° clockwise rotation sends displacement (u, v) to $(v, -u)$, this can be seen with similar triangles. So in this case $(4, 3) \mapsto (3, -4)$ and the new displacement from $(1, 1)$ is $(3, -4)$. Therefore the new apex is $(1, 1) + (3, -4) = (4, -3)$. The curve was facing left before the rotation. A direction pointing left becomes a direction pointing up after a 90° clockwise rotation.

A quadratic facing up has the form $y = (x-h)^2 + k$, where (h, k) is its apex. Since the new apex is $(4, -3)$, the resulting curve is $y = (x-4)^2 - 3 = x^2 - 8x + 13$.

Question 11

Find the sum of all distinct solutions to

$$\sin(5x + 90^\circ) = \cos 4x$$

for $0 \leq x \leq 360^\circ$, where x is measured in degrees.

A 900

B 1800

C 1440

D 1080

E 2160

Solution 11

Answer: B

Using $\sin X = \cos(90^\circ - X)$, this is an identity from GCSE maths! We get

$$\sin(5x + 90^\circ) = \cos(90^\circ - (5x + 90^\circ)).$$

So

$$\cos(-5x) = \cos 4x.$$

Since $\cos(-5x) = \cos 5x$, the equation becomes

$$\cos 5x = \cos 4x.$$

This is an equation not so different to say $\cos 5x = 0.5$. Therefore either

$$5x = 4x + 360^\circ n$$

or

$$5x = -4x + 360^\circ n.$$

From the first equation,

$$x = 360^\circ n.$$

For $0 \leq x \leq 360°$, this gives $x = 0°$ and $x = 360°$.

From the second equation,

$$9x = 360°n,$$

so

$$x = 40°n.$$

For $0 \leq x \leq 360°$, this gives

$$x = 0°, 40°, 80°, \ldots, 360°.$$

This already includes the solutions from the first equation.

These solutions form an arithmetic sequence with first term $0°$, last term $360°$, and 10 terms. Hence their sum is

$$\frac{10}{2}(0° + 360°) = 1800°.$$

Question 12

What is the smallest positive value of k for which the line $x = k$ is a line of symmetry of **both** of the curves

$$y = \sin\left(2x - \frac{\pi}{3}\right) \quad \text{and} \quad y = \cos\left(3x + \frac{\pi}{4}\right)?$$

A $\dfrac{\pi}{12}$

B $\dfrac{\pi}{4}$

C $\dfrac{5\pi}{12}$

D $\dfrac{7\pi}{12}$

E $\dfrac{11\pi}{12}$

F $\dfrac{23\pi}{12}$

Solution 12

Answer: E

For a curve of the form $y = \sin(ax + b)$ or $y = \cos(ax + b)$, a vertical line of symmetry passes through a maximum or minimum, where the function has value 1 or -1.

First curve. Since $\sin X = \pm 1$ when $X = \frac{\pi}{2} + n\pi$, we require $2x - \frac{\pi}{3} = \frac{\pi}{2} + n\pi$, where n is an integer. Hence

$$x = \frac{5\pi}{12} + \frac{n\pi}{2}.$$

Second curve. Since $\cos X = \pm 1$ when $X = m\pi$, we require $3x + \frac{\pi}{4} = m\pi$, where m is an integer. Hence

$$x = -\frac{\pi}{12} + \frac{m\pi}{3}.$$

Therefore, we need integers n and m such that

$$\frac{5\pi}{12} + \frac{n\pi}{2} = -\frac{\pi}{12} + \frac{m\pi}{3},$$

with the common value of x as small and positive as possible.

Since this is a multiple-choice question, we can check the options directly. The only option that can be written in both required forms is $\frac{11\pi}{12}$, obtained when $n = 1$ and $m = 3$.

Therefore, the smallest positive value of k is

$$\frac{11\pi}{12}.$$

Remark. Equating the two expressions for x gives $3n - 2m = -3$. The integer solutions are $n = 1 + 2t$ and $m = 3 + 3t$, where t is an integer. Substituting back gives

$$x = \frac{11\pi}{12} + t\pi.$$

The smallest positive value occurs when $t = 0$, giving $x = \frac{11\pi}{12}$. This full calculation is not needed when the options are available.

Question 13

Find the units digit of $2^{2026} + 3^{2026}$.

A 1

B 3

C 5

D 7

E 9

Solution 13

Answer: B

The units digits of powers of 2 repeat in cycles of four:

$$2, 4, 8, 6.$$

Since $2026 = 4 \cdot 506 + 2$, the units digit of 2^{2026} is the second number in the cycle, which is 4.

The units digits of powers of 3 also repeat in cycles of four:

$$3, 9, 7, 1.$$

Therefore, the units digit of 3^{2026} is the second number in this cycle, which is 9.

Hence the units digit of $2^{2026} + 3^{2026}$ is the units digit of $4 + 9 = 13$, which is 3.

Question 14

The function f is defined for all real x by

$$f(x) = \frac{1}{4}x^4 - \frac{1}{3}kx^3 - k^2x^2 + k,$$

where k is a real constant. Given that the equation $f(x) = 0$ has four distinct real roots, what are the possible values of k?

A $k > \sqrt[3]{\frac{12}{5}}$

B $k < -\sqrt[3]{\frac{12}{5}}$ or $k > \sqrt[3]{\frac{12}{5}}$

C $k > \sqrt[3]{\frac{3}{8}}$

D $k < 0$ or $k > \sqrt[3]{\frac{12}{5}}$

E $0 < k < \sqrt[3]{\frac{12}{5}}$

F $k > \sqrt{\frac{12}{5}}$

Solution 14

Answer: A

Differentiate: $f'(x) = x^3 - kx^2 - 2k^2x = x(x+k)(x-2k)$.

If $k = 0$, then $f(x) = \frac{1}{4}x^4$, so $f(x) = 0$ does not have four distinct real roots.

Now suppose $k \neq 0$. The stationary points occur at $x = -k$, $x = 0$, and $x = 2k$. They are minimum, maximum and minimum respectively. For 4 distinct roots, we must have the minimums below x-axis and maximum above x-axis.

Start with maximum at $x = 0$, $f(0) = k$, so we must have $k > 0$.

Next, both local minima must be below the x-axis.

At $x = -k$,

$$f(-k) = \frac{1}{4}k^4 + \frac{1}{3}k^4 - k^4 + k = k - \frac{5}{12}k^4,$$

and hence

$$k - \frac{5}{12}k^4 < 0.$$

Since $k > 0$, this gives

$$k^3 > \frac{12}{5}.$$

At $x = 2k$,

$$f(2k) = 4k^4 - \frac{8}{3}k^4 - 4k^4 + k = k - \frac{8}{3}k^4.$$

So

$$k - \frac{8}{3}k^4 < 0.$$

Since $k > 0$, this gives

$$k^3 > \frac{3}{8}.$$

Combining these conditions, we need

$$k^3 > \frac{12}{5}.$$

Therefore

$$k > \sqrt[3]{\frac{12}{5}}.$$

Question 15

A sequence is defined by $a_0 = \frac{1}{2}$, and $a_n = 2\sum_{r=0}^{n-1} a_r$ for $n \geq 1$. Evaluate

$$\sum_{r=1}^{\infty} \frac{1}{a_r + a_{r+1}}.$$

A the sum does not converge

B $\dfrac{3}{8}$

C $\dfrac{5}{6}$

D $\dfrac{4}{5}$

E $\dfrac{5}{9}$

Solution 15

Answer: B

The sequence (a_n) is a geometric sequence in disguise! To see this, let

$$S_n = \sum_{r=0}^{n} a_r.$$

For $n \geq 1$, we have

$$a_n = 2\sum_{r=0}^{n-1} a_r = 2S_{n-1}.$$

Therefore

$$S_n = S_{n-1} + a_n = S_{n-1} + 2S_{n-1} = 3S_{n-1}.$$

Since $S_0 = a_0 = \frac{1}{2}$, it follows that S_n is a geometric sequence:

$$S_n = \frac{1}{2} \cdot 3^n.$$

For $n \geq 1$,

$$a_n = S_n - S_{n-1} = \frac{1}{2} \cdot 3^n - \frac{1}{2} \cdot 3^{n-1} = 3^{n-1}$$

is also a geometric sequence!

Now for sequence of the summation, for $r \geq 1$,

$$\frac{1}{a_r + a_{r+1}} = \frac{1}{3^{r-1} + 3^r} = \frac{1}{4 \cdot 3^{r-1}},$$

is also geometric, with $a = 1/4$ and common ratio $= 1/3$, hence

$$\sum_{r=1}^{\infty} \frac{1}{a_r + a_{r+1}} = \frac{1/4}{1 - 1/3} = \frac{3}{8}.$$

Question 16

The set of points satisfying

$$\tan(x^2 + y^2) = 1, \qquad x^2 + y^2 < k\pi,$$

is made up of finitely many separate closed curves. For each separate closed curve, consider the region enclosed by that curve.

Given that the sum of the areas of these regions is $30\pi^2$, find the smallest possible integer value of k.

A 5

B 6

C 7

D 8

E 9

F 10

Solution 16

Answer: D

Treating $x^2 + y^2$ as a single unknown, solving gives:

$$x^2 + y^2 = \frac{\pi}{4} + n\pi \geq 0.$$

This is a circle for $n = 0, 1, 2, ...$ on the right-hand side.

Since $x^2 + y^2 < k\pi$, we need

$$\frac{\pi}{4} + n\pi < k\pi.$$

Dividing by π gives

$$n + \frac{1}{4} < k.$$

If k is a positive integer, then this means

$$n = 0, 1, 2, \ldots, k - 1.$$

So there are k circles. The area enclosed by each circle is πr^2, so the sum of the areas is

$$\pi \sum_{n=0}^{k-1} \left(\frac{\pi}{4} + n\pi\right).$$

This is the sum of an arithmetic sequence, therefore it is

$$\pi^2 \left(\frac{k}{4} + \frac{k(k-1)}{2}\right).$$

Equate to $30\pi^2$, so

$$\frac{k}{4} + \frac{k(k-1)}{2} = 30 \qquad \Longleftrightarrow \qquad (2k+15)(k-8) = 0.$$

Since k is positive,

$$k = 8.$$

The smallest integer value of k is 8.

Question 17

A rectangle has its sides parallel to the coordinate axes and lies entirely within the region enclosed by $y = 2|x-4|$ and $y = x+6$.

What is the maximum possible area of the rectangle?

You may assume that the rectangle with maximum possible area has its bottom edge meeting both arms of the graph $y = 2|x-4|$.

A $\frac{40}{3}$

B 16

C 16.5

D $\frac{49}{3}$

E 15

F $\frac{50}{3}$

G $\frac{35}{2}$

Solution 17

Answer: F

Let the bottom side of the rectangle have equation $y = a$. On the lower curve,

$$a = 2|x-4|.$$

So

$$|x-4| = \frac{a}{2},$$

giving

$$x = 4 - \frac{a}{2} = \frac{8-a}{2}$$

or

$$x = 4 + \frac{a}{2} = \frac{8+a}{2}.$$

Therefore the maximum possible width at height $y = a$ is

$$\frac{8+a}{2} - \frac{8-a}{2} = a.$$

Alternatively: the arms of $2|x-4|$ have gradient of 2 or -2. At height $y = a$, each half of the bottom edge has length $a/2$, so the full bottom edge has length a. This immediately gives the length of the bottom edge of the rectangle as a.

Now take the left bottom corner to be

$$\left(\frac{8-a}{2}, a\right).$$

The top of the rectangle is limited by the line $y = x + 6$. Substituting $x = \frac{8-a}{2}$ gives

$$y = \frac{8-a}{2} + 6 = 10 - \frac{a}{2}.$$

So the height of the rectangle is

$$10 - \frac{a}{2} - a = 10 - \frac{3a}{2}.$$

Hence the area is

$$A = a\left(10 - \frac{3a}{2}\right) = 10a - \frac{3a^2}{2}.$$

Differentiate:

$$A' = 10 - 3a.$$

For a maximum, set $A' = 0$, giving $a = \frac{10}{3}$, which is clearly a maximum.

Therefore the maximum area is

$$A = \frac{10}{3}\left(10 - \frac{3}{2} \cdot \frac{10}{3}\right) = \frac{10}{3} \cdot 5 = \frac{50}{3}.$$

Remark. More generally, suppose a finite triangular region is enclosed by a graph of the form $y = |ax + b|$ and a straight line $y = cx + d$, where the line lies above the V-shaped graph between the two intersection points. Among all rectangles with sides parallel to the coordinate axes and lying entirely inside this region, the rectangle with maximum possible area **always** has its bottom edge meeting both arms of the V-shaped graph. In particular, the maximum rectangle can never be the type which intersects only one arm of the V-shaped graph.

In our example, it can be shown that the maximum area of the other type of rectangle, which has its top-left corner on $y = x + 6$ and its bottom-right corner on the right arm of $y = 2|x-4|$, and its bottom-left is not on the left arm of $y = 2|x-4|$, is 16, which is actually only slightly smaller than $\frac{50}{3}$!

Question 18

A positive integer k is chosen uniformly at random from $1, 2, 3, \ldots, 8$.

For each non-negative integer n, define

$$a_n = (2k+1)\sum_{r=0}^{n}\binom{n}{r}\left(\frac{k-4}{4}\right)^r.$$

Let

$$S = a_0 + a_1 + a_2 + \cdots.$$

What is the probability that S converges to a finite value and $S > 8$?

A $\frac{1}{8}$

B $\frac{1}{4}$

C $\frac{3}{8}$

D $\frac{1}{2}$

E $\frac{5}{8}$

F $\frac{3}{4}$

G $\frac{7}{8}$

H 1

I 0

Solution 18

Answer: B

Remark: This question is a more challenging variation of a similar past exam question, and I felt students could benefit from more practice with this type of problem, so here it is!

By the binomial theorem,

$$\sum_{r=0}^{n}\binom{n}{r}\left(\frac{k-4}{4}\right)^r=\left(1+\frac{k-4}{4}\right)^n=\left(\frac{k}{4}\right)^n.$$

So $a_n=(2k+1)\left(\frac{k}{4}\right)^n$, and hence

$$S=(2k+1)\left(1+\frac{k}{4}+\left(\frac{k}{4}\right)^2+\cdots\right).$$

This converges when $\frac{k}{4}<1$, so $k<4$. Since k is a positive integer, the possible values are $k=1,2,3$.

For these values,

$$S=\frac{2k+1}{1-\frac{k}{4}}=\frac{4(2k+1)}{4-k}.$$

We need $S>8$, so

$$\frac{4(2k+1)}{4-k}>8.$$

Since $k<4$, we can multiply by $4-k$ without changing the inequality. This gives $4(2k+1)>8(4-k)$, so $8k+4>32-8k$, hence $16k>28$. Therefore $k>\frac{7}{4}$.

Among $k=1,2,3$, this gives $k=2,3$. Therefore there are 2 successful values out of 8, so the required probability is $\frac{1}{4}$. The answer is **B**.

Question 19

A triangle ABC has $AB = x$, $BC = 2$, and $\angle BAC = 30°$. Find the range of values of x for which there are two non-congruent possible triangles satisfying these conditions, and in both triangles $\angle ABC$ is acute.

A $0 < x < 2$

B $2 < x < 4$

C $2 < x < 2\sqrt{3}$

D $2\sqrt{3} < x < 4$

E $x > 2\sqrt{3}$

F $0 < x < 2\sqrt{3}$

Solution 19

Answer: D

Let $\angle ACB = C$. By the sine rule:

$$\frac{2}{\sin 30°} = \frac{x}{\sin C} \quad \Rightarrow \quad \sin C = \frac{x}{4}.$$

For there to be two non-congruent possible triangles, $\sin C = \frac{x}{4}$ must give two possible values of C.

Let the smaller possible value of C be θ, where $0° < \theta < 90°$. Then the two possible values of C are θ and $180° - \theta$.

For the triangle with $C = \theta$, where $0° < \theta < 90°$, we must have:

$$0 < \sin\theta < 1 \quad \Rightarrow \quad 0 < \frac{x}{4} < 1 \quad \Rightarrow \quad x < 4.$$

For the triangle with $C = 180° - \theta$, to ensure the triangle is valid, we need the angle at B to be positive:

$$B = 180° - 30° - (180° - \theta) = \theta - 30° > 0 \quad \Rightarrow \quad \theta > 30°.$$

Since $\sin\theta = \frac{x}{4}$, this gives

$$\frac{x}{4} > \frac{1}{2} \quad \Rightarrow \quad x > 2.$$

Therefore, together $2 < x < 4$.

Now impose the condition that $\angle ABC$ is acute in both triangles.

If $C = 180^\circ - \theta$, then $B = \theta - 30^\circ$, which is automatically acute when $30^\circ < \theta < 90^\circ$.

If $C = \theta$, then

$$B = 180^\circ - 30^\circ - \theta = 150^\circ - \theta.$$

For this to be acute, we need

$$150^\circ - \theta < 90^\circ,$$

so $\theta > 60^\circ$. Since $\sin\theta = \frac{x}{4}$, this gives

$$\frac{x}{4} > \frac{\sqrt{3}}{2} \quad \Rightarrow \quad x > 2\sqrt{3}.$$

Combining this with previous conditions on x, we obtain

$$2\sqrt{3} < x < 4.$$

Question 20

The function f is defined by

$$f(x) = \ln\left(\frac{1+x}{1-x}\right), \qquad -1 < x < 1.$$

Solve

$$f(x) = 4f\left(\frac{1}{\sqrt{2}}\right).$$

It may be helpful to first find constants a and b such that, for $u, v \in (-1, 1)$,

$$f\left(\frac{u+v}{1+uv}\right) = af(u) + bf(v).$$

A $\dfrac{3\sqrt{2}}{4}$

B $\dfrac{8\sqrt{2}}{9}$

C $\dfrac{4\sqrt{2}}{5}$

D $\dfrac{12\sqrt{2}}{17}$

E $\dfrac{7\sqrt{2}}{10}$

Solution 20

Answer: D

First lets find a and b. We have

$$f\left(\frac{u+v}{1+uv}\right) = \ln\left(\frac{1+\frac{u+v}{1+uv}}{1-\frac{u+v}{1+uv}}\right) = \ln\left(\frac{(1+u)(1+v)}{(1-u)(1-v)}\right).$$

So

$$f\left(\frac{u+v}{1+uv}\right) = \ln\left(\frac{1+u}{1-u}\right) + \ln\left(\frac{1+v}{1-v}\right) = f(u) + f(v).$$

Therefore $a = 1$ and $b = 1$.

Now apply this identity with $u = v = \frac{1}{\sqrt{2}}$:

$$2f\left(\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right) = f\left(\frac{\sqrt{2}}{1+\frac{1}{2}}\right) = f\left(\frac{2\sqrt{2}}{3}\right).$$

Apply it again,

$$4f\left(\frac{1}{\sqrt{2}}\right) = 2f\left(\frac{2\sqrt{2}}{3}\right) = f\left(\frac{2\cdot\frac{2\sqrt{2}}{3}}{1+\left(\frac{2\sqrt{2}}{3}\right)^2}\right) = f\left(\frac{12\sqrt{2}}{17}\right).$$

Therefore

$$f(x) = f\left(\frac{12\sqrt{2}}{17}\right).$$

Since f is one-one on $-1 < x < 1$,

$$x = \frac{12\sqrt{2}}{17}.$$