JZ Mock Set C Paper 2 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

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Question 1

Let n denote a positive integer. Consider the following three statements.

I. For every n, if n^2 is divisible by 12, then n is divisible by 6.

II. For every n, if n^2 is divisible by 12, then n is divisible by 12.

III. There exists an n such that n^2 is divisible by 12 but n is not divisible by 12.

Which of the statements are true?

A I only

B II only

C III only

D I and II only

E II and III only

F I and III only

G all of them

Solution 1

Answer: F

The notation: $a|n$ means n is divisible by a.

Statement I If n^2 is divisible by $12 = 4 \times 3$, then n^2 is divisible by both 3 and 4. Since 3 is prime, $3 \mid n^2$ forces $3 \mid n$. Also $4 \mid n^2$ forces n to be even, so $2 \mid n$. Hence n is divisible by both 2 and 3, that is by 6, so statement I is true.

Statement II Take $n = 6$, so $n^2 = 36 = 12 \times 3$ is divisible by 12, yet 6 is not divisible by 12. So II is false.

Statement III That same example makes statement III true.

So I and III hold and II fails.

Question 2

Let x, y, z be real numbers with $x \geq y$. Which of the following statements **must** be true?

1. $x^3 \geq y^3$
2. $x^2 \geq y^2$
3. $xz \geq yz$
4. $xz^2 \geq yz^2$

A none of them

B 1 only

C 4 only

D 1 and 2 only

E 1 and 3 only

F 1 and 4 only

G 2 and 4 only

H 1, 2 and 4 only

I 1, 3 and 4 only

J 1, 2, 3 and 4

Solution 2

Answer: F

Statement 1: $x^3 \geq y^3$. The function $t \mapsto t^3$ is strictly increasing on all of $\mathbb{R}$, so $x \geq y \Rightarrow x^3 \geq y^3$. **True**.

Statement 2: $x^2 \geq y^2$. Squaring is **not** monotonic on $\mathbb{R}$. Counterexample: $x = 1$, $y = -2$. Then $x \geq y$ but $x^2 = 1 < 4 = y^2$. **False**.

Statement 3: $xz \geq yz$. Multiplying an inequality by z preserves its direction only when $z \geq 0$. Counterexample: $x = 1$, $y = 0$, $z = -1$. Then $xz = -1 < 0 = yz$. **False**.

Statement 4: $xz^2 \geq yz^2$. Here the multiplier is $z^2 \geq 0$, therefore $x \geq y \Rightarrow xz^2 \geq yz^2$. **True**.

Only statements 1 and 4 must hold.

Question 3

Given that $\sin\theta + \cos\theta = \frac{1}{2}$, find the value of $\sin^3\theta + \cos^3\theta$.

A $\dfrac{1}{4}$

B $\dfrac{5}{8}$

C $\dfrac{7}{8}$

D $\dfrac{7}{12}$

E $\dfrac{11}{16}$

F $\dfrac{13}{18}$

Solution 3

Answer: E

Squaring the given equation implies $1 + 2\sin\theta\cos\theta = \frac{1}{4}$, which implies

$$\sin\theta\cos\theta = -\frac{3}{8}.$$

Then

$$\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)^3 - 3\sin\theta\cos\theta(\sin\theta + \cos\theta) = \frac{1}{8} - 3\left(-\frac{3}{8}\right)\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{9}{16} = \frac{11}{16}.$$

Question 4

Consider the following statements about the polynomial $f(x)$, where $a < b$:

(1) $f(a) > f(b)$;

(2) $f'(x) \leq 0$ for all $x \in [a, b]$;

(3) $\int_a^b f(x)\, dx \leq 0$;

(4) $f(a) + f(b) \leq 2f\left(\frac{a+b}{2}\right)$.

Which of these statements is a **necessary** condition for $f(x)$ to be decreasing for $a \leq x \leq b$? Here, decreasing means that $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$.

A none of them

B Only 1.

C Only 2.

D Only 3.

E Only 4.

F Only 1 and 2.

G Only 1 and 3.

H Only 1 and 4.

I Only 2 and 3.

J Only 2 and 4.

Solution 4

Answer: C

Statement 1 is not necessary. For example, if $f(x) = 1$, then $f(x)$ is decreasing according to the definition given, but $f(a) = f(b)$.

Statement 2 is necessary, because you may remember from A-level Mathematics that the definition of $f(x)$ being decreasing is that $f'(x) \leq 0$ throughout the interval.

This is equivalent to saying that $x_1 \le x_2$ implies $f(x_1) \ge f(x_2)$, because $f'(x) \le 0$ means that the graph never rises as x increases. Therefore, a point further to the right cannot have a greater function value.

Statement 3 is not necessary. For example, let $f(x) = 1 - x$ on $[0, 1]$. This function is decreasing, but

$$\int_0^1 (1 - x)\, dx = \frac{1}{2} > 0.$$

Statement 4 is not necessary. For example, let $f(x) = x^2$ on $[-1, 0]$. This function is decreasing, but

$$f(-1) + f(0) = 1 > 2f\left(-\frac{1}{2}\right) = \frac{1}{2}.$$

Therefore, only **statement 2** is necessary.

Question 5

Let $y(x)$ be a polynomial defined for all real values of x.

Consider the following statements:

Statement P: The function y has a local maximum at $x = 0$.

Statement Q: $y'(0) = 0$ and $y''(0) < -1$.

Which one of the following is true?

A P is neither necessary nor sufficient for Q.

B P is necessary but not sufficient for Q.

C P is sufficient but not necessary for Q.

D P is necessary and sufficient for Q.

E Whether P is necessary or sufficient for Q depends on the degree of y.

Solution 5

Answer: B

This should be a very easy question if you have been **really** paying attention in your A-level Mathematics lessons:

The conditions $y' = 0$ and $y'' < 0$ at $x = a$ are only **sufficient** for there to be a local maximum at $x = a$. Investigate $y = -x^4$, for example.

Now, $y''(0) < -1$ implies $y''(0) < 0$, so it is also a sufficient condition for a local maximum at $x = 0$. This makes P an inevitable consequence of Q, so P is necessary but not sufficient for Q.

Question 6

Let $x \neq 0$ and

$$f(x) = \frac{3}{7}x^{7/3} + \frac{3}{4}x^{4/3} - 6x^{1/3}.$$

A student attempts to find the values of x for which the curve is increasing.

Their working is shown below.

I. $f'(x) = x^{4/3} + x^{1/3} - 2x^{-2/3}$

II. $f'(x) = x^{-2/3}(x^2 + x - 2)$

III. $x^{-2/3}(x^2 + x - 2) \geq 0$

IV. $x^2 + x - 2 \geq 0$

V. $(x-1)(x+2) \geq 0$

VI. $x \leq -2$ or $x \geq 1$

Which of the following is true?

A The first error is in step **I**.

B The first error is in step **II**.

C The first error is in step **III**.

D The first error is in step **IV**.

E The first error is in step **V**.

F The first error is in step **VI**.

G The student's answer is correct.

Solution 6

Answer: G

For $x \neq 0$, differentiating term by term gives

$$f'(x) = \frac{3}{7} \cdot \frac{7}{3}x^{4/3} + \frac{3}{4} \cdot \frac{4}{3}x^{1/3} - 6 \cdot \frac{1}{3}x^{-2/3}$$

so

$$f'(x) = x^{4/3} + x^{1/3} - 2x^{-2/3}.$$

Therefore step I is correct.

Now

$$x^{4/3} = x^{-2/3}x^2 \quad \text{and} \quad x^{1/3} = x^{-2/3}x,$$

so

$$f'(x) = x^{-2/3}(x^2 + x - 2).$$

Therefore step II is correct.

The curve is increasing when $f'(x) \geq 0$, so step III is correct. Since $x \neq 0$, we have $x^{-2/3} > 0$, so dividing by $x^{-2/3}$ does not change the direction of the inequality. Hence

$$x^2 + x - 2 \geq 0.$$

Therefore step IV is correct.

Next,

$$x^2 + x - 2 = (x - 1)(x + 2),$$

so step V is correct. Finally, $(x - 1)(x + 2) \geq 0$ outside the two roots, giving

$$x \leq -2 \quad \text{or} \quad x \geq 1.$$

Therefore step VI is correct.

So the student's working is correct!

Remark: When dividing by an expression involving x, we must check carefully whether that expression could be zero. Sometimes division is valid; other times it is not!

Question 7

Six sealed black bags are labelled P, Q, R, S, T and U. Each bag contains the same number of balls. This number is a non-zero whole number from 3 to 8 inclusive.

Each bag has a label with a statement written on it. The statement may be true or false.

Bag P has a label which says: "The number of balls in this bag is a multiple of 3."

Bag Q has a label which says: "The number of balls in this bag is even."

Bag R has a label which says: "The number of balls in this bag is prime."

Bag S has a label which says: "The number of balls in this bag is at most 4."

Bag T has a label which says: "The number of balls in this bag is at least 7."

Bag U has a label which says: "The number of balls in this bag is a perfect square."

Given that exactly one of the six statements is true, which bag has the true statement on its label?

A P

B Q

C R

D S

E T

F U

Solution 7

Answer: C

This question is inspired by a similar style of question from past exams. It is a good example of a situation where a truth table is useful.

Let n be the number of balls in each bag. We simply check which labels would be true for each possible value of n.

n	True labels
3	$P,\ R,\ S$
4	$Q,\ S,\ U$
5	R
6	$P,\ Q$
7	$R,\ T$
8	$Q,\ T$

Only $n = 5$ gives exactly one true statement, where R is true, so the answer is R.

Question 8

A polynomial $p(x)$ satisfies $p(2) = 7$ and $p(-3) = -8$.

Which of the following can be deduced, where $q(x)$ denotes some polynomial?

A $p(x) = (x + 2)(x - 3)\, q(x) + 3x + 1$

B $p(x) = (x - 2)(x + 3)\, q(x) - 3x + 1$

C $p(x) = (x - 2)(x + 3)\, q(x) + 3x - 1$

D $p(x) = (x - 2)(x + 3)\, q(x) + 3x + 1$

E $p(x) = (x - 2)(x + 3)\, q(x) - x + 5$

F $p(x) = (x - 2)(x + 3)\, q(x) + x - 5$

Solution 8

Answer: D

Divide $p(x)$ by the quadratic $(x - 2)(x + 3)$. The remainder must have degree less than 2, so write

$$p(x) = (x - 2)(x + 3)\, q(x) + ax + b.$$

Applying the remainder theorem at $x = 2$ and $x = -3$:

$$p(2) = 2a + b = 7,$$
$$p(-3) = -3a + b = -8.$$

Subtracting the second from the first gives $5a = 15$, so $a = 3$, and then $b = 7 - 2(3) = 1$.

Hence $p(x) = (x - 2)(x + 3)\, q(x) + 3x + 1$.

Question 9

How many ordered pairs of integers (x, y) satisfy

$$(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)=63?$$

A 0

B 1

C 2

D 3

E 4

F 6

G 8

H infinitely many

Solution 9

Answer: E

Apply the identities $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$ in turn, or otherwise expand:

$$(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)=(x^3+y^3)(x^3-y^3)=x^6-y^6!$$

So the equation is $x^6-y^6=63$. starting from $1^6=1$, $2^6=64$, the integer sixth powers increase in value very rapidly, and difference of consecutive integer powers of 6 increases rapidly, so the only possibilty is $2^6-1^6=63$. Which, when possible signs are taken into account, yields four integer pairs: $(2,1),\ (2,-1),\ (-2,1),\ (-2,-1)$.

Remark: If you didn't know the identity, upon expanding $(x+y)(x^2-xy+y^2)$ to get x^3+y^3, you can immediately deduce $(x-y)(x^2+xy+y^2)=x^3-y^3$, because this is just the first expansion with y replaced by $-y$.

Question 10

In a TMUA-style question, students are asked to determine which, if any, of the five statements $1, 2, 3, 4, 5$ are true. They must then identify the correct option from the list provided in the question.

Assume that a **competent student** can always correctly determine the truth value of any statement they choose to check.

A teacher is designing such a question and wants to choose the option set that **maximises the smallest number of checks** a lucky student might need to determine the correct option. Equivalently, the teacher wants to maximise the fewest checks after which it is **possible** for the student to know the correct option.

Which of the following four candidate option sets best achieves the teacher's aim?

Option Set I

A: only statement 1 is true
B: only statements $1, 2, 3$ are true
C: only statements $1, 2, 4$ are true
D: only statements $1, 4, 5$ are true
E: only statements $1, 5$ are true

Option Set II

A: only statement 1 is true
B: only statement 2 is true
C: only statement 3 is true
D: only statement 4 is true
E: only statement 5 is true

Option Set III

A: only statements $1, 2$ are true
B: only statements $2, 3$ are true
C: only statements $3, 4$ are true
D: only statements $4, 5$ are true
E: no statements are true

Option Set IV

A: only statements $1, 2$ are true
B: only statements $2, 3$ are true
C: only statements $3, 4$ are true
D: only statements $4, 5$ are true
E: only statements $1, 5$ are true

A **Option Set I**

B **Option Set II**

C **Option Set III**

D **Option Set IV**

Solution 10

Answer: D

For each option set, we look for the smallest number of statement checks that could identify an option in the luckiest case.

If one statement check can uniquely identify an option, then the smallest possible number of checks for that option set is 1.

For **Option Set I**, statement 3 is true only in option B. So if a student checks statement 3 and finds it true, they immediately know the answer is B. Therefore the lucky minimum is 1.

For **Option Set II**, each option says exactly one different statement is true. So if the student checks the true statement, they immediately know the answer. Therefore the lucky minimum is 1.

For **Option Set III**, statement 1 is true only in option A, and statement 5 is true only in option D. So one lucky check is enough. Therefore the lucky minimum is 1.

For **Option Set IV**, each statement appears in exactly two options. For example, statement 1 is true in A and E, statement 2 is true in A and B, and so on. Therefore no single statement check can ever identify the correct option. Therefore the minimum number of checks is greater than 1, and this is our best choice!

It can be shown that the lucky minimum for Option Set IV is 2, though this detail is not required to deduce the answer to the question.

Thus the teacher should choose **Option Set IV**, since it maximises the smallest number of checks a lucky student might need.

Remark: Even setting TMUA choices can be a logical puzzle!

Question 11

In triangle ABC, $\angle A = 30^\circ$, $AB = 2x^2$ and $BC = 3x$, where $x > 0$. Suppose that there are exactly two non-congruent triangles satisfying these conditions. Which of the following conditions can be **deduced**?

A $2 < x < \frac{7}{2}$

B $\frac{5}{3} < x < 5$

C $1 < x < 4$

D $\frac{1}{2} < x < \frac{5}{2}$

E $0 < x < 2$

Solution 11

Answer: C

For two non-congruent triangles to be possible, the side BC must lie strictly between the perpendicular height and the side AB.

The perpendicular height is

$$AB \sin 30^\circ = 2x^2 \cdot \frac{1}{2} = x^2.$$

Therefore, two non-congruent triangles are possible if and only if

$$x^2 < 3x < 2x^2.$$

Since $x > 0$, the first inequality gives

$$x < 3,$$

and the second inequality gives

$$x > \frac{3}{2}.$$

Hence

$$\frac{3}{2} < x < 3.$$

Looking at all the options, the only conclusion that can be drawn is: $1 < x < 4$.

Question 12

A student makes the following **claim**:

If f is a real-valued function such that, for some constant a, $\dfrac{f(x)+f(-x)}{2} = a$ for all real values of x, then there exists a constant k such that $f(x) = kx + a$ for all real values of x.

Examine their **claim** above, and determine which of the following is true?

A The **claim** is true.

B $f(x) = 2x + 1$ is a counterexample to the claim.

C $f(x) = \sin x$ is a counterexample to the claim.

D $f(x) = \tan x$ is a counterexample to the claim.

E $f(x) = \cos x$ is a counterexample to the claim.

F $f(x) = -x$ is a counterexample to the claim.

Solution 12

Answer: C

A counterexample must satisfy the assumption of the if, but fail the then conclusion.

For (C), if $f(x) = \sin x$, then

$$\frac{f(x)+f(-x)}{2} = \frac{\sin x + \sin(-x)}{2} = 0.$$

So the assumption is true with $a = 0$. However, $\sin x$ is cleary not of the form $kx + 0$, so (C) is a counterexample, and this statement is correct, so it is the answer.

For completeness, let's check the rest.

For (B), if $f(x) = 2x + 1$, then

$$\frac{f(x)+f(-x)}{2} = \frac{(2x+1)+(-2x+1)}{2} = 1.$$

So the assumption is true with $a = 1$. However, $f(x) = 2x + 1$ is of the form $kx + a$, with $k = 2$ and $a = 1$. So (B) is not a counterexample.

For (D), $f(x) = \tan x$ is not defined for all real values of x, for example at $x = \frac{\pi}{2}$. Therefore it does not satisfy the assumption of the claim, so it is not a counterexample.

For (E), if $f(x) = \cos x$, then

$$\frac{f(x) + f(-x)}{2} = \frac{\cos x + \cos x}{2} = \cos x.$$

This is not constant, so the assumption is false. Hence (E) is not a counterexample.

For (F), if $f(x) = -x$, then

$$\frac{f(x) + f(-x)}{2} = \frac{-x + x}{2} = 0.$$

So the assumption is true with $a = 0$. However, $f(x) = -x$ is of the form $kx + a$, with $k = -1$ and $a = 0$. So (F) is not a counterexample.

Question 13

Let x and y be non-zero real numbers with $x \neq y$.

Which one of the following statements is both **necessary and sufficient** for $x < y$?

A $y^{-1/3} < x^{-1/3}$

B $y^{1/5} < x^{1/5}$

C $x^{2/3} < y^{2/3}$

D $y^{2/3} < x^{2/3}$

E $|x|^{1/3} < |y|^{1/3}$

F $x^{-1/3} < y^{-1/3}$

G $x^{1/5} < y^{1/5}$

Solution 13

Answer: G

Here are some general facts to note.

If n is odd, then $a < b$ is equivalent to $a^{1/n} < b^{1/n}$, since taking an odd nth root preserves the inequality.

However, if n is even, then $a < b$ is not equivalent to $a^n < b^n$, since raising both sides to an even power does not preserve the inequality in general.

Apply the first result to our $x < y$, we can see that $x^{1/5} < y^{1/5}$ is definitely equivalent, and given there is only one equivalent statement, this must be it.

Remark: It is possible to rule out each of the distractors with a single numerical counterexample.

For **A**, take $x = -1$ and $y = 1$. Then $x < y$, but

$$y^{-1/3} = 1 \quad \text{and} \quad x^{-1/3} = -1,$$

so $y^{-1/3} < x^{-1/3}$ is false.

For **B**, take $x = 1$ and $y = 2$. Then $x < y$, but

$$2^{1/5} < 1^{1/5}$$

is false.

For **C**, take $x = -8$ and $y = -1$. Then $x < y$, but

$$x^{2/3} = 4 \quad \text{and} \quad y^{2/3} = 1,$$

so $x^{2/3} < y^{2/3}$ is false.

For **D**, take $x = 1$ and $y = 8$. Then $x < y$, but

$$y^{2/3} = 4 \quad \text{and} \quad x^{2/3} = 1,$$

so $y^{2/3} < x^{2/3}$ is false.

For **E**, take $x = -8$ and $y = -1$. Then $x < y$, but

$$|x|^{1/3} = 2 \quad \text{and} \quad |y|^{1/3} = 1,$$

so $|x|^{1/3} < |y|^{1/3}$ is false.

For **F**, take $x = 1$ and $y = 8$. Then $x < y$, but

$$x^{-1/3} = 1 \quad \text{and} \quad y^{-1/3} = \frac{1}{2},$$

so $x^{-1/3} < y^{-1/3}$ is false.

Question 14

For which of the following is it a **necessary but not sufficient** condition that

$$2x^3 - 3px^2 + p = 0$$

has exactly one real root?

A $-1 < p < 1$

B $-1 \le p \le 1$

C $0 < p < 1$

D $p = 0$

E $p \le 0$

F $p \ge 0$

Solution 14

Answer: B

Let $f(x) = 2x^3 - 3px^2 + p$, then $f'(x) = 6x^2 - 6px = 6x(x - p)$. So the stationary points are at $x = 0$ and $x = p$.

Their corresponding values are $f(0) = p$ and $f(p) = 2p^3 - 3p^3 + p = p - p^3 = p(1 - p^2)$.

First consider $p \neq 0$. For the cubic to have exactly one real root, the two stationary values must have the same sign. If either stationary value is zero, then the cubic has a repeated root and another real root. So we need

$$f(0)f(p) = p \cdot p(1 - p^2) > 0.$$

This gives

$$p^2(1 - p^2) > 0.$$

Since $p^2 > 0$, we need

$$1 - p^2 > 0.$$

Hence

$$-1 < p < 0 \qquad \text{or} \qquad 0 < p < 1.$$

Now we check $p = 0$ separately. In this case, $f(x) = 2x^3$, which has exactly one real root, namely $x = 0$.

Therefore, the necessary and sufficient condition is $-1 < p < 1$.

Option A is necessary and sufficient, so it is not the answer.

Option B is necessary because $-1 < p < 1$ implies $-1 \leq p \leq 1$, but the converse is not true. Therefore, it is a necessary but not sufficient condition.

So, Option B is the correct answer.

Remark: Since we are looking for a necessary but not sufficient condition, and there should be only one correct option, we can immediately rule out A because A implies B. If A were the solution, then B would also be a solution, but there can be only one solution. In the same way, we can immediately rule out C and D. However, in reality, these eliminations do not make it any quicker to deduce the correct option for this question!

Question 15

Which one of the following numbers is largest in value?

A $\log_2 7$

B $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$

C $\dfrac{1}{\sqrt{3}-\sqrt{2}}$

D $\sqrt{12}$

E $\sqrt{6+4\sqrt{2}}$

Solution 15

Answer: D

A: $\log_2 7 < \log_2 8 = 3$.

B: Geometric series with first term 1 and ratio 2/3: $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1-2/3} = 3$.

C: Rationalise:

$$\frac{1}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \sqrt{3}+\sqrt{2}.$$

D: $\sqrt{12} = 2\sqrt{3}$.

E: Recognise $6+4\sqrt{2} = 4+4\sqrt{2}+2 = (2+\sqrt{2})^2$, so $\sqrt{6+4\sqrt{2}} = 2+\sqrt{2}$.

It is now clear that the largest must be one of D or E. Suppose, for contradiction, that D is smaller than E. Since both sides are positive,

$$\begin{aligned} 2\sqrt{3} &< 2+\sqrt{2} \\ 12 &< 6+4\sqrt{2} \\ 6 &< 4\sqrt{2} \\ 3 &< 2\sqrt{2} \\ 9 &< 8. \end{aligned}$$

This is impossible, so the assumption that D is smaller than E is false. Therefore,

$$2\sqrt{3} > 2+\sqrt{2}.$$

Hence D is the largest number.

Question 16

Three transformations of the plane are defined as follows. R is the reflection in the y-axis. T is the translation by 4 units in the positive x-direction. S is the horizontal stretch with scale factor 2 parallel to the x-axis, fixing the y-axis.

Let $y = f(x)$ be a function defined on the real numbers.

When the graph of $y = f(x)$ is transformed by applying R, then T, then S in that order, the result is the graph of $y = g(x)$.

When the graph of the same function $y = f(x)$ is transformed by applying S, then T, then R in that order, the result is the graph of $y = h(x)$.

Which one of the following conditions on $y = f(x)$ is **necessary and sufficient** for the functions $g(x)$ and $h(x)$ to be identical?

A $f(x) = f(x + 3)$ for all x

B $f(x) = f(x + 4)$ for all x

C $f(x) = f(x + 6)$ for all x

D $f(x) = f(x + 12)$ for all x

E $f(x) = f(-x)$ for all x

F $f(x) = f(6 - x)$ for all x

Solution 16

Answer: C

Compute $g(x)$ by applying R, T, and S in turn to $y = f(x)$.

Apply R by replacing x with $-x$: $y = f(-x)$.

Apply T by replacing x with $x - 4$: $y = f(-(x - 4)) = f(4 - x)$.

Apply S by replacing x with $\frac{x}{2}$: $y = f\left(4 - \frac{x}{2}\right)$.

Therefore, $g(x) = f\left(4 - \frac{x}{2}\right)$.

Now compute $h(x)$ by applying S, T, and R in turn to $y = f(x)$.

Apply S: $y = f\left(\frac{x}{2}\right)$.

Apply T: $y = f\left(\frac{x-4}{2}\right) = f\left(\frac{x}{2} - 2\right)$.

Apply R: $y = f\left(-\frac{x}{2} - 2\right)$.

Therefore, $h(x) = f\left(-\frac{x}{2} - 2\right)$.

For $g(x) = h(x)$ for all x, we need

$$f\left(4 - \frac{x}{2}\right) = f\left(-\frac{x}{2} - 2\right)$$

for all x.

Let $u = -\frac{x}{2} - 2$. As x ranges over $\mathbb{R}$, so does u, and $4 - \frac{x}{2} = u + 6$. Hence the condition is

$$f(u+6) = f(u) \text{ or } f(x) = f(x+6).$$

Remark: This question is inspired by a similar exam question, but is a more difficult version. It tests students' understanding of the order and application of horizontal transformations, which is often subtly misunderstood.

Question 17

Determine the total number of ordered pairs of positive integers (x, y) satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{9}.$$

A 5

B 6

C 8

D 9

E 10

F 12

Solution 17

Answer: A

Clearing denominators gives $9(x + y) = xy$, so $xy - 9x - 9y = 0$.

Now, **crucially**, notice that we want to factorise this in some way so that we can enumerate the possible values of x and y. This can be done by adding a constant to both sides. If $x = 9$, for example, then the xy and $9y$ terms cancel, leaving -81. Hence the correct constant to add is 81, which completes the factorisation, as this makes $(x - 9)$ a factor:

$$xy - 9x - 9y + 81 = 81 \quad \Longrightarrow \quad (x - 9)(y - 9) = 81.$$

Also, since $1/y > 0$, we have $1/x < 1/9$, so $x > 9$. Similarly, $y > 9$. Therefore, both $x - 9$ and $y - 9$ must be positive divisors of 81.

Each positive divisor d of 81 gives exactly one solution

$$x = 9 + d, \qquad y = 9 + \frac{81}{d},$$

and both values are automatically positive integers.

Since

$$81 = 3^4,$$

the number of positive divisors of 81 is

$$4 + 1 = 5.$$

Therefore, there are 5 ordered pairs.

Remark: This method of factorising an expression involving two variables is extremely useful for TMUA questions. It is covered, or will be covered, in the **Additional Useful Topics** section of my site. The question itself belongs to another well-known class of solvable Diophantine equations.

Question 18

Let $f(x) = x^3 - px^2 - p^2x + k$ where k is a real constant. Suppose that p and q are real numbers satisfying $-3 \leq p \leq 3$ and $-3 \leq q \leq 3$.

The function f has non-negative gradient at $x = q$.

Find the area of the region of all possible points (p, q) in the p-q plane.

A 20

B 21

C 24

D 30

E 36

F 42

G 56

H 60

Solution 18

Answer: C

We have $f'(x) = 3x^2 - 2px - p^2$. Since f has non-negative gradient at $x = q$, we need

$$3q^2 - 2pq - p^2 \geq 0.$$

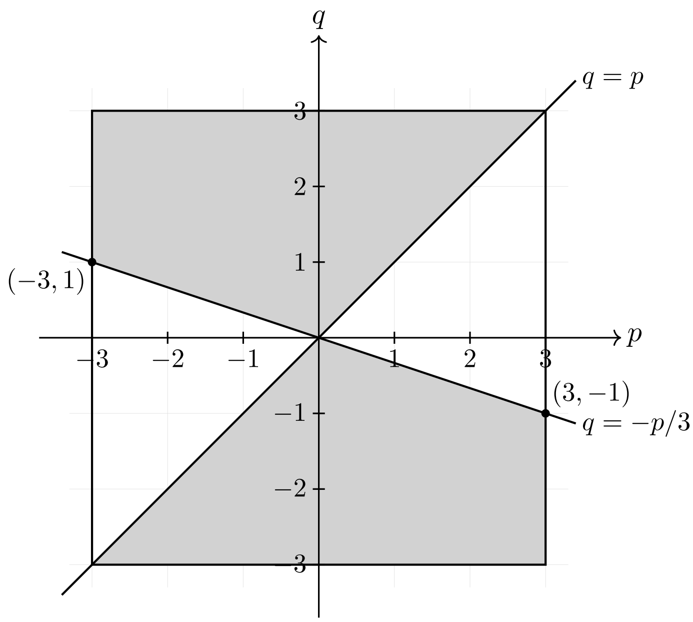
Notice that p is a root, so $(q - p)$ is a factor, and factorising gives

$$3q^2 - 2pq - p^2 = (3q + p)(q - p).$$

So we need

$$(3q + p)(q - p) \geq 0.$$

Therefore $q - p \geq 0$ and $3q + p \geq 0$, or $q - p \leq 0$ and $3q + p \leq 0$. See the diagram below.



The regions are congruent when split at $p = 0$, so we can just work out the right side, and then double the area.

The entire right half of the square is a rectangle of area $3 \times 6 = 18$.

The unshaded region between the two lines is a triangle. Its vertical side lies on $p = 3$, between $(3, -1)$ and $(3, 3)$, so its base has length 4. Its perpendicular height is 3. Hence its area is $\frac{1}{2} \times 4 \times 3 = 6$.

Therefore, the shaded area for $p \geq 0$ is $18 - 6 = 12$, and the total for the region is twice of this which is 24.

Question 19

An infinite sequence of integers $u_1, u_2, u_3, \ldots$ satisfies $u_1 = 10$ and

$$u_n + 2 < u_{n-1} < u_n + 9 \quad \text{for } n > 1.$$

It is given that $u_k = -50$ and that there exists some integer j with $1 < j < k$ such that $u_j = -10$.

How many different values can k take?

Disclaimer: This question is inspired by a question from tylertutoring.com. I found the original question very interesting and useful, so I created a new version with a couple of additional twists. Perhaps a few too many?! Enjoy!

A 4

B 9

C 10

D 12

E 13

F 15

G 36

H 48

Solution 19

Answer: D

First trick to overcome: the awkward nature of the inequality. We note that it can be rewritten as $u_{n-1} - 8 \leq u_n \leq u_{n-1} - 3$. This form is much more helpful, as we can see that to get to the next term, you just subtract an integer between 3 to 8 inclusive from the previous term.

Let's first consider what are the possible steps to get to -10 from 10. Then, the least number of steps is 3 (subtract as fast as possible, 8, to reach -10), and greatest is 6 (subtract as slow as possible). Let the number of steps to -10 be x, then x is a number from 3 to 6 inclusive.

Similarly, next consider the possible steps to get from -10 to -50. The least number of steps is 5 (5 steps of 8), and the most is 13 (12 steps of 3, last step of 4). Let y be the number of steps from -10 to -50, then y is a number from 5 to 13 inclusive.

k is the term count from term 1 then add x and add y steps, so $k = x + y + 1$. Therefore it is enough to count the number of possible values of $x + y$. For examples, smallest $x + y$ is $3 + 5 = 8$, that is to say you cannot get from 10 to -10 then to -50 any faster than 8 steps. Likewise, the greateset $x + y$ is $6 + 13 = 19$.

Therefore possible values of $x + y$ are 8, 9, 10,...,19, so a total of $19 - 7 = 12$ values, and hence there are also 12 possible values for k. The answer is **D**.

Question 20

A quadrilateral is convex if each of its interior angles is less than $180°$.

Let $ABCD$ be a convex quadrilateral, with its vertices labelled in order. A quadrilateral is called a rhombus if all four of its sides are equal in length.

Which of the following statements, taken individually, are **sufficient** to guarantee that $ABCD$ is a rhombus?

1. $AB = AD$, and the diagonals AC and BD intersect at right angles.

2. $AB = CD$, and the diagonals AC and BD divide $ABCD$ into four similar triangles.

3. The diagonals AC and BD divide $ABCD$ into four similar triangles.

4. The diagonal AC divides $ABCD$ into two triangles, and the diagonal BD divides $ABCD$ into two triangles, with all four of these triangles having the same area.

5. $AB = BC = CD$, and the triangles ABC and BCD are congruent.

A Only 1.

B Only 2.

C Only 4.

D Only 5.

E Only 1 and 3.

F Only 2 and 3.

G Only 2 and 4.

H Only 3 and 4.

I Only 5 and 1.

J Only 2 and 5.

K Only 1 and 2.

L None of them.

Solution 20

Answer: B

Disclaimer: Sorry! This question is longer than I originally intended. I had planned to include only three statements, but all five turned out to be interesting, so I was unwilling to delete any of them. In any case, for students this means more practice, and therefore more learning!

Statement 1 is not sufficient. For example, take

$$A = (1, 0), \quad B = (0, 1), \quad C = (-2, 0), \quad D = (0, -1).$$

Then the diagonals AC and BD are perpendicular, and $AB = AD = \sqrt{2}$. However, $BC = CD = \sqrt{5}$, so $ABCD$ is not a rhombus.

Statement 2 is sufficient, although this is quite hard to see!

Let the diagonals meet at O, and write

$$OA = a, \quad OB = b, \quad OC = c, \quad OD = d.$$

Since the four triangles AOB, BOC, COD and DOA are all similar, the triangles AOB and BOC are similar. The angles $\angle AOB$ and $\angle BOC$ are supplementary.

If two similar non-degenerate triangles have supplementary angles at O, those angles must both be $90°$. (Proof of this crucial fact by contradiction is left as an exercise!) Hence the diagonals are perpendicular.

So the four small triangles are similar right-angled triangles.

Now compare triangles AOB and COD. They are similar right-angled triangles. Also, statement 2 gives $AB = CD$, which are their hypotenuses. Therefore the two triangles are congruent.

Hence either $a = c$ and $b = d$, or $a = d$ and $b = c$.

If $a = c$ and $b = d$, then

$$AB = BC = CD = DA = \sqrt{a^2 + b^2}.$$

So $ABCD$ is a rhombus.

If $a = d$ and $b = c$, then triangle BOC has equal perpendicular sides. Since all four small triangles are similar, all four are isosceles right-angled triangles. Hence again all four sides of $ABCD$ are equal.

So statement 2 is sufficient.

Statement 3 is a weaker version of 2 and is not sufficient. It can describe a kite that is not a rhombus. For example, take two congruent 3-4-5 right-angled triangles and join them along their common hypotenuse, such that the sides with length 3 are joined and the sides with length 4 are joined. The resulting shape is a kite which is not a rhombus, and it has the property that its diagonals divide it into four similar triangles.

Statement 4 is not sufficient. A non-square rectangle satisfies statement 4, since each diagonal divides the rectangle into two triangles of equal area. However, a non-square rectangle is not a rhombus.

Statement 5 is not sufficient. For example, take $A = (-2, \sqrt{3})$, $B = (-1, 0)$, $C = (1, 0)$ and $D = (2, \sqrt{3})$. These points form a convex quadrilateral, and $AB = BC = CD = 2$.

Also, triangles ABC and BCD are congruent by SSS, since $AB = CD$, BC is common, and $AC = BD = 2\sqrt{3}$. However, $AD = 4$, so $ABCD$ is not a rhombus.

Therefore the only sufficient statement is **statement 2**.