JZ Mock Set D Paper 2 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

Last updated: 5 September 2026 at 23:13

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Question 1

How many distinct real roots does the equation $x^4 - 4x^3 + 4x^2 - 1 = 0$ have?

A 0

B 1

C 2

D 3

E 4

Solution 1

Answer: D

Notice that $x^4 - 4x^3 + 4x^2 = (x^2 - 2x)^2$, so the equation becomes $(x^2 - 2x)^2 = 1$. Hence $x^2 - 2x = 1$ or $x^2 - 2x = -1$.

The first equation, $x^2 - 2x - 1 = 0$, has discriminant $8 > 0$, so it has two distinct real roots. Since 8 is not a square, both roots are irrational. The second equation, $x^2 - 2x + 1 = (x-1)^2 = 0$, has the single root 1, which is rational and therefore cannot equal either root of the first equation. Hence there are 3 distinct real roots.

Question 2

Real numbers w, x, y, z all satisfy $w, x, y, z > 1$ and

$$\log_w x = x, \qquad \log_x y = y, \qquad \log_y z = z.$$

Which of the following is equal to $\log_w z$?

A xyz

B $\dfrac{1}{xyz}$

C $\dfrac{1}{wxyz}$

D $\dfrac{1}{x+y+z}$

E $\dfrac{1}{xy}$

F $\dfrac{1}{yz}$

G $x + y + z$

H $wxyz$

Solution 2

Answer: A

Convert each given equation to exponential form and chain the substitutions.

From $\log_w x = x$ we have $x = w^x$.

From $\log_x y = y$ we have $y = x^y = (w^x)^y = w^{xy}$, so $\log_w y = xy$.

From $\log_y z = z$ we have $z = y^z = (w^{xy})^z = w^{xyz}$, so $\log_w z = xyz$.

Question 3

Given $\tan\theta = 2,\ 90^\circ < \theta < 360^\circ$, find the value of $\sin\theta\cos\theta$.

A $\dfrac{2}{5}$

B $-\dfrac{2}{5}$

C $-\dfrac{5}{3}$

D $\dfrac{3}{5}$

E $-\dfrac{1}{5}$

Solution 3

Answer: A

Solution. This question can be solved directly using the definitions of the trigonometric functions, by considering a point P moving around a circle of radius r.

Since $\tan\theta = 2 > 0$ and $90^\circ < \theta < 360^\circ$, the angle must lie in the third quadrant. By definition,

$$\tan\theta = \frac{y}{x} = 2,$$

where (x, y) are the coordinates of P. To keep the numbers simple, take $x = -1$ and $y = -2$, since both coordinates must be negative in the third quadrant.

Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1+4} = \sqrt{5}.$$

Therefore,

$$\sin\theta\cos\theta = \frac{y}{r}\cdot\frac{x}{r} = \left(-\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}\right) = \frac{2}{5}.$$

Question 4

Five riders, Archimedes, Boole, Cantor, Descartes and Euler, are competing in a race. Whenever a rider changes position, the other riders keep their relative order.

Early in the race, Euler is ahead of Descartes, and Cantor is also ahead of Descartes. After a tight corner, Cantor drops back exactly one place. Later, Archimedes drops to last place. On the final lap, Descartes drops back exactly three places, and there are no further changes.

What is the final order of the riders, from first to fifth?

A Euler, Cantor, Boole, Archimedes, Descartes.

B Euler, Boole, Cantor, Archimedes, Descartes.

C Cantor, Euler, Boole, Archimedes, Descartes.

D Euler, Cantor, Archimedes, Boole, Descartes.

E Euler, Descartes, Cantor, Boole, Archimedes.

F Cantor, Boole, Euler, Archimedes, Descartes.

Solution 4

Answer: A

Immediately before Descartes drops back three places, he must be in either first or second place. Since Euler was ahead of Descartes earlier, and none of the previous changes can reverse the relative order of Euler and Descartes, Euler is still ahead of Descartes. Hence Descartes cannot be first, so Descartes is second and Euler is first.

Cantor was initially ahead of Descartes. If Cantor had not dropped past Descartes when he dropped back exactly one place, then Cantor would still be ahead of Descartes immediately before the final lap. But Euler is already ahead of Descartes, so Descartes could not then be second. Therefore Cantor must have been immediately ahead of Descartes and dropped back past him. Hence Cantor is immediately behind Descartes.

Archimedes has already dropped to last place, so the order immediately before the final change is

Euler, Descartes, Cantor, Boole, Archimedes.

Descartes then drops back exactly three places, giving the final order

Euler, Cantor, Boole, Archimedes, Descartes.

Therefore the correct answer is A.

Question 5

Triangle ABC is right-angled at C. The perpendicular from C to the hypotenuse AB meets AB at D.

The perimeters of triangles ACD and BCD are 18 and 24 respectively. Find the area of triangle ABC.

A $\frac{65}{2}$

B 30

C $\frac{149}{4}$

D 32

E $\frac{75}{2}$

F 40

Solution 5

Answer: E

Drawing the perpendicular from the right angle to the hypotenuse produces two triangles that are both similar to the original triangle.

Since triangles ACD and BCD are similar to triangle ABC, their perimeters are in the same ratio as their hypotenuses. Therefore,

$$AC : BC = 18 : 24 = 3 : 4.$$

Hence triangle ABC has side lengths in the ratio $3 : 4 : 5$. In fact, all three triangles are scaled versions of the $(3, 4, 5)$ right-angled triangle.

The hypotenuse of triangle ACD, which is AC, can be found as

$$18 \div (3 + 4 + 5) \times 5 = \frac{5}{2} \times 3.$$

Since AC is the shortest side of ABC, this means that the outer triangle ABC is a scaled version of the $(3, 4, 5)$ right-angled triangle with scale factor $\frac{5}{2}$.

The area of the $(3, 4, 5)$ right-angled triangle is 6, so triangle ABC has area

$$6\left(\frac{5}{2}\right)^2 = \frac{75}{2}.$$

Question 6

Which of the following is a **sufficient** condition on the real constant k for the equation

$$(x+k)^{1/2} + (x-k)^{1/4} = k$$

to have at least one real solution for x?

A $k = 0$ or $k > 3$

B $k = 1$ or $k > 3$

C $k = 0$ or $1 < k < 3$

D $k \geq 1$

E $0 \leq k \leq 3$

Solution 6

Answer: A

Since the left-hand side is non-negative, there can be no solution when $k < 0$. When $k = 0$, the equation has the solution $x = 0$.

Now suppose $k > 0$. The domain requires $x \geq k$. The left-hand side is increasing in x, so its least value occurs at $x = k$ and is $(2k)^{1/2}$. Hence a solution exists if and only if $(2k)^{1/2} \leq k$. Since $k > 0$, this is equivalent to $2k \leq k^2$, so $k \geq 2$.

Therefore the equation has a real solution if and only if $k = 0$ or $k \geq 2$. Of the given conditions, $k = 0$ or $k > 3$ is sufficient.

Question 7

Which of the following is a **necessary but not sufficient** condition for

$$x^4 - 4x^2 + c = 0$$

to have exactly 4 distinct real roots?

A $-2 < c < 3$

B $-1 < c < 5$

C $1 < c < 6$

D $-3 < c < 2$

E $2 < c < 7$

Solution 7

Answer: B

Let $y = x^2$. Then the equation becomes

$$y^2 - 4y + c = 0,$$

so $y = 2 \pm \sqrt{4 - c}$.

For the original equation to have four distinct real roots, these must be two distinct positive values of y. They are real and distinct when $c < 4$, while the smaller root is positive when $2 - \sqrt{4 - c} > 0$, which gives $c > 0$.

Hence the equation has exactly four distinct real roots if and only if $0 < c < 4$.

A necessary but not sufficient condition must contain every value in $(0, 4)$, as well as some values outside this interval. The condition $-1 < c < 5$ does this. It is not sufficient since, for example, $c = 4.5$ satisfies $-1 < c < 5$ but does not give four distinct real roots.

Therefore, the answer is $-1 < c < 5$.

Question 8

Let a be a real number with $a > 1$, and let f be a function such that

$$f(0) = 1, \qquad f(2) > a^2, \qquad f(-1) < a^{-1}.$$

Consider the following four statements.

I. $f(x) = b^x$ for some real number b with $b > a$.

II. $f(x) = b^x$ for some real number b with $0 < b < a$.

III. $f(x) = a^{kx}$ for some real number k.

IV. $f(x) = b^{x^2}$ for some positive real number b.

Which of the statements **might** be true?

A Statement I only.

B Statement III only.

C Statements I and III only.

D Statements I, II and III only.

E Statements I, III and IV only.

F Statements II and IV only.

G Statements II, III and IV only.

H All four statements.

Solution 8

Answer: C

For statement I, if $f(x) = b^x$ with $b > a$, then

$$f(0) = 1, \qquad f(2) = b^2 > a^2, \qquad f(-1) = \frac{1}{b} < \frac{1}{a}.$$

Therefore, statement I might be true.

For statement II, $0 < b < a$ gives $f(2) = b^2 < a^2$, so statement II cannot be true.

For statement III, choosing $k > 1$ gives $a^k > a$, so $f(x) = a^{kx} = (a^k)^x$ satisfies all three conditions. Therefore, statement III might be true.

For statement IV, $f(2) > a^2$ would give $b^4 > a^2$, so $b > \sqrt{a} > 1$. However, $f(-1) = b < a^{-1} < 1$, which is impossible, so IV cannot be true.

Hence statements I and III only might be true.

Question 9

In triangle ABC, $AB = t$, $BC = 10$ and $\angle BAC = 45°$, where $t > 0$.

For which values of t is the length of AC uniquely determined?

A $0 < t \leq 10$

B $0 < t < 10\sqrt{2}$

C $0 < t \leq 10$ or $t = 10\sqrt{2}$

D $t = 10\sqrt{2}$

E $t > 10$

F $0 < t \leq 10\sqrt{2}$

Solution 9

Answer: C

The length AC is uniquely determined if and only if the information determines a unique triangle.

There are two ways this can happen.

First, ABC may be right-angled at C. In this case, AB is the hypotenuse, so

$$\sin 45° = \frac{BC}{AB} = \frac{10}{t}.$$

Hence $t = 10\sqrt{2}$.

Second, we may have $AB \leq BC$, so $t \leq 10$. Since the larger side is opposite the larger angle, this gives $\angle ACB \leq \angle BAC = 45°$. The supplementary possibility for $\angle ACB$ would therefore be at least $135°$, which is impossible since $\angle BAC = 45°$. Thus there is only one possible triangle.

Therefore the required values of t are given by: $0 < t \leq 10$ or $t = 10\sqrt{2}$.

Question 10

Let f be a function with domain $\{1, 2, 3, 4\}$ and range contained in $\{1, 2, 3, 4\}$. For how many such functions is

$$f(f(x)) = x$$

for every x in the domain of f?

A 1

B 4

C 7

D 9

E 10

F 15

Solution 10

Answer: E

First, let's show that the range must be $\{1, 2, 3, 4\}$. Suppose not, and that some $a \in \{1, 2, 3, 4\}$ is not in the range of f. Taking $x = a$ in $f(f(x)) = x$ gives $f(f(a)) = a$. Since $f(a)$ is an element of the domain, $f(f(a))$ is an output of f. Thus a is in the range of f, a contradiction. Hence the range must be exactly $\{1, 2, 3, 4\}$.

Also, each element must either map to itself, or be interchanged with one other element. For example, if $f(1) = 2$, then $f(f(1)) = 1$ gives $f(2) = 1$.

There are three cases.

Case 1: all four elements map to themselves, there is 1 function.

Case 2: exactly one pair is interchanged, there are $\binom{4}{2} = 6$ choices for that pair.

Case 3: two pairs are interchanged, choose the element paired with the number 1, there are clearly 3 ways, after which the remaining two elements must be paired.

Therefore the total number of functions is $1 + 6 + 3 = 10$.

Question 11

In how many ways can 8 identical balls be placed into 3 distinct boxes so that no box contains more than 4 balls?

A 21

B 30

C 45

D 10

E 15

F 24

Solution 11

Answer: E

For now, let us simplify the problem and remove the restriction that no box may contain more than 4 balls. What is the total number of ways to distribute the balls? How do we count them?

Imagine the 8 balls laid out in a row. To split them into 3 groups, we can add 2 dividers to divide the row of 8 balls into 3 groups. Therefore, we can think of the problem as having 10 positions, of which we need to choose 2 for the dividers, with the rest occupied by the balls. Note that if we choose positions 1 and 2 for the dividers, then the distribution of balls is $0, 0, 8$, which is allowed in the unrestricted problem. Therefore, the total number is given by 10 choose 2:

$$\binom{10}{2} = 45.$$

Now let us subtract the cases in which a box has 5 or more balls, to bring back our restriction. Suppose the first box contains at least 5 balls. Place 5 balls in that box first, leaving 3 balls to distribute freely among the three boxes, including the box with 5 balls. So now we need to place 2 dividers among a row of 3 balls. This is the same problem, but with fewer balls, so the total number of ways of doing this is

$$\binom{5}{2} = 10.$$

There are 3 choices for the box containing at least 5 balls, so that gives $10 \times 3 = 30$ ways. Also, no distribution has two such boxes, since that would require at least 10 balls.

Therefore, the required number of ways is $45 - 30 = 15$.

Alternatively, we can enumerate them in an orderly way, writing the numbers of balls in the three boxes as a three-digit numbers, in ascending order:

$$044,\ 134,\ 143,\ 224,\ 233,\ 242,\ 314,\ 323,\ 332,\ 341,\ 404,\ 413,\ 422,\ 431,\ 440.$$

Here, for example, 134 means that the three boxes contain 1, 3 and 4 balls respectively. The initial zero in 044 means that the first box is empty.

Now count, there are 15 of them.

Question 12

A TMUA question asks candidates to identify which one of the conditions below is a **sufficient but not necessary** condition for some statement P, concerning a real number x, to hold.

Friedrich Gauss, sitting the exam, noticed to his delight that the examiner has designed the question poorly, and the correct option can be deduced without knowing or examining P in the slightest!

Which is the correct option?

A $|x-2| < 2$

B $x^2 - 5x + 6 < 0$

C $\log(x-2)$ is defined

D $|x-2| < 1$

E x can be any real number

Solution 12

Answer: B

Let a condition be called correct if it is sufficient but not necessary for P.

Suppose condition Q is correct, and condition R implies Q. Then

$$R \implies Q \implies P,$$

so R is also sufficient for P.

Also, since Q is not necessary for P, there is some value of x for which P is true but Q is false. Since $R \implies Q$, this same value also makes R false. Hence R is also not necessary for P.

Therefore, if one option is implied by another option, the weaker option cannot be the **unique** correct answer.

Now rewrite the options:

$$|x-2| < 2 \iff 0 < x < 4,$$

$$x^2 - 5x + 6 < 0 \iff (x-2)(x-3) < 0 \iff 2 < x < 3,$$

$$\log(x-2) \text{ is defined} \iff x > 2,$$

$$|x-2| < 1 \iff 1 < x < 3.$$

The condition $2 < x < 3$ implies each of the following:

$$0 < x < 4, \qquad x > 2, \qquad 1 < x < 3, \qquad x \in \mathbb{R}.$$

So if any of those four conditions were correct, then $2 < x < 3$ would also be correct. That would give more than one correct option, which is impossible.

Hence all the other options are eliminated, and the only possible correct option is **B**.

Question 13

The real numbers x, y and z are such that exactly two of the following five statements are true.

Which two statements are true?

I $\quad x \leq y + z$

II $\quad x \leq |y| + |z|$

III $\quad y + z < |x|$

IV $\quad x + y + z > 0$

V $\quad x + y + z \leq 0$

A Statements I and II.

B Statements I and IV.

C Statements I and V.

D Statements III and IV.

E Statements II and III.

F Statements II and IV.

G Statements II and V.

H Statements III and V.

Solution 13

Answer: D

Statements IV and V are negations of each other, so exactly one of them is true. Therefore, exactly one of statements I, II and III is true.

Statement I cannot be the only true one, because $y + z \leq |y| + |z|$, so

$$x \leq y + z \leq |y| + |z|.$$

Thus statement II would also be true.

Statement II cannot be the only true one. If III is false, then $y + z \geq |x| \geq x$, so statement I is also true.

Therefore, III is the only true statement among I, II and III. In particular, II is false, so

$$x > |y| + |z|.$$

Hence $x > 0$, and moving $|y|$ and $|z|$ to side of x gives:

$$x + y + z \geq x - |y| - |z| > 0.$$

Therefore, IV is true and V is false.

Hence the two true statements are III and IV.

For example, $x = 3$, $y = 1$ and $z = 1$ satisfy these conditions.

Question 14

For a real number x, let $\{x\}$ denote the fractional part of x. For example, $\{1.2\} = 0.2$ and $\{-1.5\} = 0.5$.

A sequence (a_n) is defined by $a_1 = \frac{1}{2}$ and

$$a_{n+1} = a_n \int_0^1 |4\{nx\} - 2| \; dx$$

for each positive integer n.

What is the value of

$$\sum_{n=1}^{\infty} a_n?$$

A $\quad 1$

B $\quad 2$

C $\quad 4$

D $\quad 12$

E $\quad \dfrac{9}{2}$

F $\quad \infty$

Solution 14

Answer: F

As x runs from 0 to 1, the fractional part $\{nx\}$ repeatedly increases from 0 to 1, completing this pattern exactly n times.

Hence the graph of $|4\{nx\} - 2|$ consists of n identical V-shaped sections. Each section has width $\frac{1}{n}$ and height 2, so its area is

$$\frac{1}{2} \times \frac{1}{n} \times 2 = \frac{1}{n}.$$

Therefore,

$$\int_0^1 |4\{nx\} - 2| \; dx = n\left(\frac{1}{n}\right) = 1.$$

The recurrence becomes $a_{n+1} = a_n$. Since $a_1 = \frac{1}{2}$, we have $a_n = \frac{1}{2}$ for every positive integer n. Therefore the terms do not tend to 0, so the infinite series $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

Question 15

A sequence is defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for every positive integer n. How many of the terms $a_1, a_2, \ldots, a_{100}$ are divisible by 4?

A 8

B 16

C 12

D 5

E 15

F 20

G 24

Solution 15

Answer: B

Only the remainders on division by 4 matter, since the remainder of a sum **depends only** on the remainders of the two numbers being added. Listing the remainders of $a_1, a_2, a_3, \ldots$ gives

$$1,\ 1,\ 2,\ 3,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 1,\ 0,\ \ldots$$

At $n = 7, 8$ the pair of remainders is $1, 1$, which is exactly the pair we started with. Since each term is determined by the previous two, the list of remainders repeats with period 6 from the start.

Within each block the remainder 0 occurs once, in the sixth position. So a_n is divisible by 4 exactly when n is divisible by 6, and these values of n form the arithmetic sequence $6, 12, 18, \ldots, 96$. That sequence has $96 \div 6 = 16$ terms.

Question 16

You are an assistant at JZ Maths, tasked with helping to set a TMUA-style multiple-choice question.

The question contains five statements, labelled $(1), (2), (3), (4), (5)$, and the student must determine exactly which statements are true. You, as the question setter, know that exactly two of the five statements are true, but this information will not be given to the student.

Each answer option is a possible set of true statements. For example, the option $\{1, 2, 5\}$ means that statements $(1), (2), (5)$ only are true, while the option $\{\}$ means that none of the statements are true.

Under Policy 1 of JZ Maths, the question must be made **as difficult as possible**: even a competent student who can correctly check every statement must determine the truth value of all five statements in order to guarantee identifying the correct option. In particular, checking any four statements must not be sufficient to determine the correct option.

Under Policy 2 of JZ Maths, the question must contain **as few answer options** as possible, so that it **appears easier** than it really is.

How many answer options should you set in order to comply with both policies? Failure to comply will result in your immediate dismissal.

Disclaimer. Unrelated to this question, please be aware that no such policies officially exist at JZ Maths of course! There may be a good reason for this, as you will learn later.

A 4

B 5

C 6

D 7

E 8

F 9

G 10

Solution 16

Answer: C

Suppose, without loss of generality, that the two true statements are (1) and (2), so the correct option is $\{1, 2\}$.

If the student checks statements $(2), (3), (4), (5)$ but not (1), there must still be another possible option which agrees on those four statements. The only such option is $\{2\}$.

Similarly, omitting the check of statement (2) requires the option $\{1\}$. Omitting statement (3) requires $\{1, 2, 3\}$, and likewise statements (4) and (5) require $\{1, 2, 4\}$ and $\{1, 2, 5\}$.

Thus the options must include

$$\{1,2\}, \quad \{2\}, \quad \{1\}, \quad \{1,2,3\}, \quad \{1,2,4\}, \quad \{1,2,5\}.$$

These six options are all distinct, and they ensure that checking any four statements is insufficient. Therefore, you should set 6 answer options.

Remark 1: The question has now passed quality control: minimal appearance, maximal inconvenience.

Remark 2: As it turns out, minimising the options may not be such a good idea after all. See the next Paper 2!

Question 17

Let f be a function defined on the real numbers.

When the graph of $y = f(x)$ is reflected in the y-axis and then stretched in the x-direction by a factor of $\frac{1}{2}$, the result is the graph of $y = g(x)$.

When the graph of $y = f(x)$ is stretched in the x-direction by a factor of $\frac{1}{2}$ and then translated by 1 unit in the negative x-direction, the result is the graph of $y = h(x)$.

Which of the following is a **sufficient but not necessary condition** for $g(x)$ and $h(x)$ to be identical, that is $g(x) = h(x)$ for all real values of x?

A $f(0) = f(2)$

B

$$f(x) = f(2 - x) \text{ for all } x$$

C

$$f(x + 1) = f(1 - x) \text{ for all } x$$

D

$$f(x) = \sum_{k=1}^{10} \cos(\pi k x)$$

E

$$f(x) = f(1 - x) \text{ for all } x$$

F

$$f(x) = \cos(\pi x) + \sin(\pi x)$$

Solution 17

Answer: D

Remark: This is really a question about vertical line of symmetry.

Reflecting $y = f(x)$ in the y-axis gives $y = f(-x)$, and then stretching in the x-direction by a factor of $\frac{1}{2}$ gives $g(x) = f(-2x)$.

Stretching $y = f(x)$ in the x-direction by a factor of $\frac{1}{2}$ first gives $y = f(2x)$. Translating this graph by 1 unit in the negative x-direction means replacing x by $x+1$, so

$$h(x) = f(2(x+1)) = f(2x+2).$$

Hence $g(x) = h(x)$ for all x if and only if $f(-2x) = f(2x+2)$ for all x. Writing $u = -2x$, this is equivalent to

$$f(u) = f(2-u) \text{ for all } u.$$

Thus the required condition is that the graph of f has **symmetry** about the line $x = 1$.

Option (B) is exactly this condition, so it is necessary and sufficient. Option (C) is equivalent to it: setting $u = x+1$ gives $1 - x = 2 - u$. Hence neither is sufficient but not necessary.

Option (A) checks only one pair of points, so it is not sufficient. Option (E) gives symmetry about $x = \frac{1}{2}$, not about $x = 1$. This leaves options (D) and (F).

The graph of $y = \cos x$ has lines of symmetry at $x = n\pi$ for every integer n. Therefore $y = \cos(\pi kx)$ has lines of symmetry at $x = \frac{n}{k}$. In particular, taking $n = k$ shows that every term $\cos(\pi kx)$ has symmetry about $x = 1$. Their sum therefore also has symmetry about $x = 1$, so option (D) is sufficient.

However, $y = \sin(\pi x)$ does not have symmetry about $x = 1$. Indeed, $\sin(\pi(2-x)) = -\sin(\pi x)$, so option (F) is not sufficient.

Option (D) gives one specific function satisfying the general condition $f(x) = f(2-x)$. It is therefore sufficient, but it is not necessary, since many other functions also satisfy this symmetry condition.

Therefore the answer is option (D).

Question 18

Find all values of k such that the equation

$$\int_0^y (x + 1 - k - |x - 1| + |x - k|)\, dx = 0$$

has exactly three distinct solutions for y, where $k > 1$ and $y \geq 0$.

A $k > 2$

B $k > 3$

C $k > 4$

D $k > 5$

E $3 < k < 4$

F $2 < k < 4$

G no such values of k exist

Solution 18

Answer: B

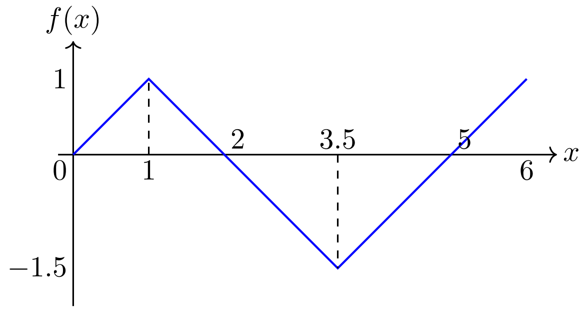
Let

$$f(x) = x + 1 - k - |x - 1| + |x - k|.$$

Resolving the modulus signs gives

$$f(x) = \begin{cases} x & \text{for } 0 \leq x < 1, \\ 2 - x & \text{for } 1 \leq x < k, \\ x + 2 - 2k & \text{for } x \geq k. \end{cases}$$

Thus the graph consists of straight-line segments with gradients 1, -1 and 1. The graph below illustrates the case $k = 3.5$.



From the diagram, it is immediately clear that $y = 0$ is a solution, another solution lies between 2 and 5, and a third lies to the right of 6. From the same example, we see that the existence of three solutions for y depends only on whether the area of the negative triangle is greater than the area of the first positive triangle. From the graph, this happens precisely when $k > 3$, so this is the required solution.

Remark: Although it is possible to evaluate the integral directly and use the resulting expression to deduce the answer, this is unwise, as it involves a large amount of calculation.

Question 19

Let f be a function defined on the real numbers. Consider the following seven statements, each intended to hold for all real values of a:

(1) $f(a) > 0$ if $a > 0$.

(2) $a > 0$ is necessary for $f(a) > 0$.

(3) $a \leq 0$ only if $f(a) \leq 0$.

(4) If $f(a) \leq 0$, then $a \leq 0$.

(5) $f(a) > 0$ whenever $a \leq 0$.

(6) $a > 0$ is sufficient for $f(a) > 0$.

(7) $f(a) \leq 0$ is sufficient and necessary for $a > 0$.

Given that **exactly one** of these seven statements is true for the particular function f in question, which one is it?

A Statement (1)

B Statement (2)

C Statement (3)

D Statement (4)

E Statement (5)

F Statement (6)

G Statement (7)

Solution 19

Answer: E

Remark. We had to include this classic somewhere for a little more practice, so here it is.

We first reformulate each statement equivalently, placing the condition involving a first.

Statement (1) : $a > 0 \quad \Rightarrow \quad f(a) > 0.$

Statement (2) : $a \leq 0 \quad \Rightarrow \quad f(a) \leq 0.$

Statement (3) : $a \leq 0 \quad \Rightarrow \quad f(a) \leq 0.$

Statement (4) : $a > 0 \quad \Rightarrow \quad f(a) > 0.$

Statement (5) : $a \leq 0 \quad \Rightarrow \quad f(a) > 0.$

Statement (6) : $a > 0 \quad \Rightarrow \quad f(a) > 0.$

Statement (7) : $a > 0 \quad \Leftrightarrow \quad f(a) \leq 0.$

Statements (1), (4) and (6) are equivalent, so none can be the only true statement. Similarly, statements (2) and (3) are equivalent, so neither can be the only true statement.

Also, statement (7) implies statement (5), since $a \leq 0$ forces $f(a) > 0$. Therefore statement (7) cannot be the only true statement.

Hence the only possible answer is Statement (5).

Question 20

Let

$$y = ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h, \qquad a \neq 0,$$

where a, b, c, d, e, f, g, h are real numbers. Call a point on the graph an **inflection with zero gradient** if $\frac{dy}{dx} = 0$ there but $\frac{dy}{dx}$ does not change sign as x passes through it.

Which one of the following triples (number of local minima, number of local maxima, number of inflections with zero gradient) **is possible** for some choice of the real coefficients?

You may find it useful to recall that $\frac{dy}{dx}$ changes sign at a local maximum or minimum, but does not change sign at a stationary point of inflection.

A (0, 0, 4)

B (2, 2, 2)

C (2, 3, 0)

D (3, 3, 1)

E (1, 1, 3)

F (2, 1, 1)

G (2, 2, 1)

Solution 20

Answer: G

Since y is a polynomial of degree 7, its derivative $\frac{dy}{dx}$ is a polynomial of degree 6.

Consider the graph of $\frac{dy}{dx}$. Its roots are the points where it meets the x-axis.

At a local maximum or minimum of y, $\frac{dy}{dx}$ changes sign, so its graph crosses the x-axis. In the factorised form of $\frac{dy}{dx}$, the corresponding factor must therefore be $(x - r)^m$, where m is **odd** and r is the x-coordinate of the point.

At an inflection with zero gradient, $\frac{dy}{dx}$ does not change sign, so its graph touches the x-axis and turns back. The corresponding factor must therefore be $(x - r)^m$, where m is **even** and r is the x-coordinate of the point.

Hence each local maximum or minimum uses at least one root of $\frac{dy}{dx}$, while each inflection with zero gradient uses at least two.

Also, since $\frac{dy}{dx}$ has even degree, it has the same sign as $x \to -\infty$ and as $x \to \infty$. Therefore the number of changes from positive to negative must equal the number of changes from negative to positive, so the number of local maxima must equal the number of local minima.

Now check the options.

$(0, 0, 4)$ requires at least $2 \cdot 4 = 8$ roots of $\frac{dy}{dx}$, impossible.

$(2, 2, 2)$ requires at least $2 + 2 + 2 \cdot 2 = 8$ roots of $\frac{dy}{dx}$, impossible.

$(2, 3, 0)$ has unequal numbers of local minima and local maxima, impossible.

$(3, 3, 1)$ requires at least $3 + 3 + 2 = 8$ roots of $\frac{dy}{dx}$, impossible.

$(1, 1, 3)$ requires at least $1 + 1 + 2 \cdot 3 = 8$ roots of $\frac{dy}{dx}$, impossible.

$(2, 1, 1)$ has unequal numbers of local minima and local maxima, impossible.

But $(2, 2, 1)$ requires at least

$$2 + 2 + 2 = 6$$

roots of $\frac{dy}{dx}$, which is possible for a degree 6 derivative.

For example, $\frac{dy}{dx}$ could have four simple roots causing two local minima and two local maxima, and one double root giving one inflection with zero gradient.

So the possible triple is $(2, 2, 1)$.