JZ Mock Set E Paper 1 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

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Question 1

Find all real values of k for which

$$kx^2 + (k+3)x + (k+3) > 0$$

holds for every real number x.

A $k > 1$

B $k < -3$ or $k > 1$

C $-3 < k < 1$

D $k > 0$

E $k > -3$

F There are no such values of k

Solution 1

Answer: A

Case 1. If $k = 0$, the expression becomes $3x + 3$, which is negative when $x < -1$. Hence $k = 0$ does not work.

Case 2. Suppose $k \neq 0$. For the quadratic to be strictly positive for every real x, its leading coefficient must be positive and its discriminant must be negative. Therefore,

$$k > 0 \quad \text{and} \quad (k+3)^2 - 4k(k+3) < 0.$$

The discriminant condition simplifies to

$$(k+3)(3-3k) < 0,$$

so $(k+3)(k-1) > 0$. Hence $k < -3$ or $k > 1$.

Combining this with $k > 0$ gives $k > 1$.

Question 2

For $0 < a < 2$, let $f(x) = |x - a|$. The trapezium rule with two equal strips is used to estimate

$$\int_0^2 f(x)\,dx.$$

Find the product of all values of a for which the trapezium estimate exceed the exact integral by $\frac{1}{4}$.

A $\dfrac{4}{3}$

B $\dfrac{3}{4}$

C 2

D 1

E $\dfrac{3}{2}$

F $\dfrac{2}{3}$

Solution 2

Answer: B

The two strips have width 1, so the trapezium estimate is

$$T = \frac{1}{2}\left(f(0) + 2f(1) + f(2)\right).$$

Since $0 < a < 2$, we have $f(0) = a$, $f(1) = |1 - a|$ and $f(2) = 2 - a$. Hence $T = 1 + |1 - a|$.

The exact integral is the sum of the areas of two right-angled triangles:

$$\int_0^2 |x - a|\,dx = \frac{a^2}{2} + \frac{(2-a)^2}{2} = 1 + (a-1)^2.$$

Therefore the amount by which the trapezium estimate exceeds the exact integral is $|1-a| - (a-1)^2$.

Let $t = |1 - a|$. Since $(a-1)^2 = t^2$, we require

$$t - t^2 = \frac{1}{4},$$

so $(t - \frac{1}{2})^2 = 0$. Thus $|1 - a| = \frac{1}{2}$, giving $a = \frac{1}{2}$ or $a = \frac{3}{2}$.

Their product is $\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$.

Remark: Alternatively, after $T = 1 + |1 - a|$, split into the cases $a < 1$ and $a \geq 1$ to remove the modulus.

Question 3

The points $A = (0, 0)$ and $B = (6, 0)$ are given, and a point P lies on the line $y = x + 2$. Find the minimum possible value of $PA + PB$.

A $3\sqrt{10}$

B 9

C 8

D $2\sqrt{15}$

E $2\sqrt{17}$

Solution 3

Answer: E

Both A and B lie on the same side of the line $y = x + 2$. Reflect A in the line to obtain $A' = (-2, 2)$. For any point P on the line, $PA = PA'$, so $PA + PB = PA' + PB$, which is smallest when P lies on the straight segment $A'B$. The minimum value is then

$$A'B = \sqrt{(6 - (-2))^2 + (0 - 2)^2} = \sqrt{64 + 4} = 2\sqrt{17}.$$

Question 4

For how many integer values of c does the line $y = 2x + c$ meet the circle $x^2 + y^2 = 20$ at two distinct points?

A 19

B 18

C 17

D 16

E 21

F 24

Solution 4

Answer: A

Write the line as $2x - y + c = 0$. Its distance from the centre $(0, 0)$ is $\dfrac{|c|}{\sqrt{5}}$ by similar triangles. Two distinct intersection points occur exactly when the distance is less than the radius $2\sqrt{5}$:

$$\frac{|c|}{\sqrt{5}} < 2\sqrt{5} \iff |c| < 10.$$

The integers with $|c| < 10$ are $-9, -8, \ldots, 9$, of which there are 19.

Question 5

What is the value of

$$\int_{-2}^{3} \big| |x| - 1 \big| \, dx \, ?$$

A 3

B $\frac{9}{2}$

C 4

D $\frac{7}{2}$

E $\frac{13}{4}$

F 5

Solution 5

Answer: D

The expression changes form at $x = -1$, $x = 0$ and $x = 1$. Hence

$$\big||x| - 1\big| = \begin{cases} -x - 1, & -2 \le x \le -1, \\ x + 1, & -1 \le x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ x - 1, & 1 \le x \le 3. \end{cases}$$

Therefore,

$$\int_{-2}^{3} \big||x| - 1\big| \, dx = \int_{-2}^{-1} (-x - 1) \, dx + \int_{-1}^{0} (x + 1) \, dx + \int_{0}^{1} (1 - x) \, dx + \int_{1}^{3} (x - 1) \, dx.$$

These four integrals are the areas of triangles, so no need to actually integrate, just find triangular areas, giving

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 2 = \frac{7}{2}.$$

Question 6

The real number x satisfies both

$$\frac{x+3}{x^2-4} \leq 0 \quad \text{and} \quad \frac{x^2-9}{x-1} > 0.$$

Find the total length of the set of possible values of x.

A 0

B 1

C 2

D 3

E 4

F 6

G 8

H ∞

Solution 6

Answer: D

First consider

$$\frac{x+3}{x^2-4} = \frac{x+3}{(x-2)(x+2)} \leq 0.$$

The critical values are -3, -2 and 2. A sign analysis gives

$$x \leq -3 \quad \text{or} \quad -2 < x < 2.$$

Next,

$$\frac{x^2-9}{x-1} = \frac{(x-3)(x+3)}{x-1} > 0.$$

The critical values are -3, 1 and 3. A sign analysis gives

$$-3 < x < 1 \quad \text{or} \quad x > 3.$$

Both inequalities must hold, so we take the intersection:

$$((-\infty, -3] \cup (-2, 2)) \cap ((-3, 1) \cup (3, \infty)) = (-2, 1).$$

Therefore the complete set of values of x is $-2 < x < 1$, and the length of the interval is 3.

Question 7

The curve C is defined by

$$x = y + \sqrt{y} + 1.$$

Find the equation of the tangent to C at the point where $x = 3$.

A $\quad y = \frac{2}{3}x - 1$

B $\quad y = \frac{3}{2}x - \frac{7}{2}$

C $\quad y = \frac{1}{2}x - \frac{1}{2}$

D $\quad y = \frac{3}{4}x - \frac{5}{4}$

E $\quad y = -\frac{2}{3}x + 3$

Solution 7

Answer: A

Just interchange x and y, it becomes a standard question, just remember to change it back at the end!

Interchange x and y. The resulting curve is

$$y = x + \sqrt{x} + 1.$$

The point on the original curve with $x = 3$ satisfies

$$3 = y + \sqrt{y} + 1.$$

Hence $y + \sqrt{y} = 2$, so $y = 1$. Therefore, after interchanging x and y, the corresponding point is $(1, 3)$.

For the swapped curve,

$$\frac{dy}{dx} = 1 + \frac{1}{2\sqrt{x}}.$$

At $x = 1$,

$$\frac{dy}{dx} = \frac{3}{2}.$$

Therefore, the tangent to the swapped curve is

$$y - 3 = \frac{3}{2}(x - 1).$$

Interchanging x and y back again gives

$$x - 3 = \frac{3}{2}(y - 1).$$

Making y the subject,

$$y - 1 = \frac{2}{3}(x - 3),$$

so

$$y = \frac{2}{3}x - 1.$$

Question 8

How many square divisors greater than 1 does 810000 have?

A square divisor of an integer n is a factor of n which is also a square number.

A 24

B 26

C 27

D 30

E 15

F 18

Solution 8

Answer: B

First factorise 810000:

$$810000 = 81 \cdot 10000 = 3^4 \cdot 2^4 \cdot 5^4 = 2^4 \cdot 3^4 \cdot 5^4.$$

A square divisor must have an even exponent for each prime.

For each of the primes 2, 3 and 5, the exponent can be 0, 2 or 4, giving 3 choices.

Hence the number of square divisors is

$$3 \cdot 3 \cdot 3 = 27.$$

This includes 1, so the number of square divisors greater than 1 is

$$27 - 1 = 26.$$

Question 9

The sequence u_n is defined by $u_n = \frac{n}{2^{n-1}}$ for $n \geq 1$. For a positive integer n, let

$$T_n = \sum_{k=n}^{\infty} u_k.$$

Which of the following is an expression for T_n?

A

$$\frac{3n+5}{2^n}$$

B

$$\frac{7n+9}{2^{n+1}}$$

C

$$\frac{9n+7}{2^{n+1}}$$

D

$$\frac{n+1}{2^{n-2}}$$

E

$$\frac{5n+3}{2^n}$$

F

$$\frac{n+3}{2^{n-1}}$$

Solution 9

Answer: D

Remark: This solution adapts the standard method for summing a geometric series: multiply the sum by the common ratio, then subtract the two expressions. Many similar problems are solved this way.

Write

$$T_n = \frac{n}{2^{n-1}} + \frac{n+1}{2^n} + \frac{n+2}{2^{n+1}} + \cdots .$$

Halving each term gives

$$\frac{1}{2}T_n = \frac{n}{2^n} + \frac{n+1}{2^{n+1}} + \frac{n+2}{2^{n+2}} + \cdots .$$

Subtracting, we obtain:

$$\frac{1}{2}T_n = \frac{n}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots.$$

The remaining terms form an infinite geometric series with first term $\frac{1}{2^n}$ and common ratio $\frac{1}{2}$, so their sum is $\frac{1}{2^{n-1}}$. Therefore

$$\frac{1}{2}T_n = \frac{n}{2^{n-1}} + \frac{1}{2^{n-1}} = \frac{n+1}{2^{n-1}}.$$

So,

$$T_n = \frac{n+1}{2^{n-2}}.$$

Question 10

For each real number a, let $F(a)$ be the area between the graph $y = |x - a|$ and the x-axis from $x = 0$ to $x = 2$. How many real values of a satisfy

$$F(a) = a + 1?$$

A 0

B 1

C 2

D 3

E 4

Solution 10

Answer: C

If $a < 0$, then

$$F(a) = \int_0^2 (x - a)\,dx = 2 - 2a.$$

The equation $2 - 2a = a + 1$ gives $a = \frac{1}{3}$, but $a < 0$, so not valid.

If $0 \le a \le 2$, then

$$F(a) = \frac{a^2}{2} + \frac{(2-a)^2}{2} = a^2 - 2a + 2.$$

So

$$a^2 - 2a + 2 = a + 1,$$

which gives

$$a^2 - 3a + 1 = 0.$$

Of its two roots, only $\frac{3-\sqrt{5}}{2}$ lies in $[0, 2]$.

If $a > 2$, then

$$F(a) = \int_0^2 (a - x)\,dx = 2a - 2.$$

The equation $2a - 2 = a + 1$ gives $a = 3$, which is valid. Hence there are 2 solutions.

Question 11

Triangle ABC is right-angled at C, with

$$AC = 5, \qquad BC = 12, \qquad AB = 13.$$

The perpendicular from C to AB is drawn. Of the two smaller triangles formed, the larger triangle is selected.

Within the selected triangle, the perpendicular from its right-angled vertex to its hypotenuse is drawn, and the larger of the two new triangles is selected. This process is continued indefinitely.

Find the sum of the lengths of all the perpendiculars drawn.

A 60

B 30

C 48

D 80

E 72

F The sum does not converge.

Solution 11

Answer: A

Remark: This question uses a remarkable property of right-angled triangles: the perpendicular from the right angle to the hypotenuse divides the triangle into two smaller triangles, each of which is similar to the original triangle.

The first perpendicular divides triangle ABC into two triangles similar to the original. Their hypotenuses have lengths 5 and 12, while the hypotenuse of the original triangle has length 13. Therefore the larger new triangle is a scaled copy of the original, with scale factor

$$\frac{12}{13}.$$

The same reasoning applies at every stage, so each new perpendicular is $\frac{12}{13}$ times the previous one.

Let the first perpendicular have length h. Comparing two expressions for the area of triangle ABC gives

$$\frac{1}{2}(5)(12) = \frac{1}{2}(13)h,$$

so $h = \frac{60}{13}$.

The perpendicular lengths therefore form the geometric sequence

$$\frac{60}{13}, \quad \frac{60}{13}\left(\frac{12}{13}\right), \quad \frac{60}{13}\left(\frac{12}{13}\right)^2, \quad \ldots$$

Hence their sum is

$$\frac{\frac{60}{13}}{1-\frac{12}{13}} = 60.$$

Question 12

Find the sum of all positive values of k satisfying

$$\int_0^k (|2x-6|-2)\ dx = 3.$$

A 1

B 4

C 8

D 9

E 13

F 16

Solution 12

Answer: D

Sketch a graph of $y = |2x-6|-2$, the first triangle has area 4, the second which is under the x-axis has area of 2.

By inspection, it is possible to deduce the possible values of k are 1, 3 and 5, so their sum is $1+3+5=9$.

Remark: Although the question can be solved using integration, this approach is lengthy and tedious, which is the intended trap.

Question 13

The values of x, measured in degrees, satisfy both

$$\sin 3x + 2\cos 3x = \frac{\sqrt{3}+2}{2}$$

and

$$3\sin 3x - 2\cos 3x = \frac{3\sqrt{3}-2}{2}.$$

The sum of all such values of x in the interval $0 \le x \le k^\circ$ is 800°.

Find the least possible value of k.

A 240°

B 500°

C 380°

D 400°

E 360°

F 720°

Solution 13

Answer: C

We first routinely solve for $\sin 3x$ and $\cos 3x$, adding the two equations gives $\sin 3x = \frac{\sqrt{3}}{2}$, then ubstituting this into the first equation gives $\cos 3x = \frac{1}{2}$.

Remember here x has to satisfy both equations!

The general solutions of the first equation are

$$3x = 60^\circ + 360^\circ n$$

or

$$3x = 120^\circ + 360^\circ n.$$

The general solutions of the second equation are

$$3x = 60^\circ + 360^\circ m$$

or

$$3x = 300^\circ + 360^\circ m,$$

where m and n are integers.

The only values common to both sets are

$$3x = 60^\circ + 360^\circ n.$$

Hence $x = 20^\circ + 120^\circ n$. Since $x \geq 0$, the values are

$$20^\circ,\ 140^\circ,\ 260^\circ,\ 380^\circ, \ldots$$

The first four have sum $20 + 140 + 260 + 380 = 800$, so the least possible value of k is 380.

Question 14

Real numbers a, b, c satisfy

$$5^a = 2^b = 10^{-c}, \qquad abc \neq 0.$$

What is the value of $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$?

A -1

B 0

C 1

D $\dfrac{1}{2}$

E 2

Solution 14

Answer: B

Let the common value be k, so

$$5^a = 2^b = 10^{-c} = k.$$

Taking logarithms gives

$$a \ln 5 = b \ln 2 = -c \ln 10 = \ln k.$$

Since $abc \neq 0$, we have $k \neq 1$, so $\ln k \neq 0$. Hence

$$\frac{1}{a} = \frac{\ln 5}{\ln k}, \qquad \frac{1}{b} = \frac{\ln 2}{\ln k}, \qquad \frac{1}{c} = -\frac{\ln 10}{\ln k}.$$

Therefore,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{\ln 5 + \ln 2 - \ln 10}{\ln k} = \frac{\ln\left(\frac{5 \cdot 2}{10}\right)}{\ln k} = 0.$$

Question 15

You may use the following result: for any triangle with area A, perimeter p and inradius r, where the inradius is the radius of the circle which touches all three sides of the triangle,

$$A = \frac{rp}{2}.$$

A triangle has integer side lengths which form a non-constant arithmetic sequence. Its inradius is 3 and its area is 54.

What is the length of its shortest side?

A 7

B 8

C 9

D 10

E 11

Solution 15

Answer: C

Since $A = \frac{rp}{2}$, and $r = 3$ and $A = 54$, the perimeter is 36.

Let the side lengths be $m - d$, m and $m + d$, where d is a positive integer. Their sum is $3m$, so $3m = 36$ and hence $m = 12$.

Thus the side lengths are $12 - d$, 12 and $12 + d$. Let C be the angle between the sides of lengths $12 - d$ and 12, so the side opposite C has length $12 + d$.

Now we must remember that we still need to include the area constraint. Using the area formula,

$$54 = \frac{1}{2} \cdot 12(12 - d) \sin C,$$

so

$$\sin C = \frac{9}{12 - d}.$$

By the cosine rule,

$$\cos C = \frac{(12 - d)^2 + 12^2 - (12 + d)^2}{2 \cdot 12(12 - d)} = \frac{6 - 2d}{12 - d}.$$

Using $\sin^2 C + \cos^2 C = 1$ gives

$$\frac{9^2 + (6-2d)^2}{(12-d)^2} = 1.$$

Therefore

$$81 + (6-2d)^2 = (12-d)^2,$$

which simplifies to $3d^2 = 27$. Hence $d = 3$.

The side lengths are therefore 9, 12 and 15, so the shortest side has length 9.

Alternatively, from $12 - d$, 12 and $12 + d$.

By Heron's nearly 2000 year old formula:

$$A^2 = s(s-a)(s-b)(s-c).$$

Therefore,

$$54^2 = 18\big(18 - (12-d)\big)(18-12)\big(18 - (12+d)\big),$$

so

$$2916 = 18(6+d) \cdot 6(6-d) = 108(36 - d^2).$$

Hence $36 - d^2 = 27$, so $d^2 = 9$ and therefore $d = 3$.

Question 16

The function

$$f(x) = \frac{x}{x^2 + x + 1}$$

is defined for all real x. Find the difference between its maximum and minimum values.

A $\frac{4}{3}$

B $\frac{2}{3}$

C $\frac{5}{4}$

D $\frac{3}{2}$

E 3

Solution 16

Answer: A

Let $y = f(x)$. Since $x^2 + x + 1 > 0$ for all x, we can write $y(x^2 + x + 1) = x$, that is

$$yx^2 + (y - 1)x + y = 0.$$

As a quadratic in x, for a real solution x to exist, the quadratic in x needs non-negative discriminant:

$$(y - 1)^2 - 4y^2 \geq 0 \quad \Rightarrow \quad (3y - 1)(y + 1) \leq 0,$$

so $-1 \leq y \leq \frac{1}{3}$. The maximum value is $\frac{1}{3}$ and the minimum is -1, and the difference is $\frac{4}{3}$.

Question 17

Positive real numbers a, b and c satisfy $a+b+c=6$. Find the maximum value of a^2bc.

You may find the following inequality for positive real numbers x and y useful:

$$xy \le \left(\frac{x+y}{2}\right)^2,$$

with equality if and only if $x=y$.

A 20

B 24

C 36

D $\frac{81}{4}$

E 18

Solution 17

Answer: D

Applying the given inequality to b and c gives

$$bc \le \left(\frac{b+c}{2}\right)^2 = \left(\frac{6-a}{2}\right)^2.$$

Therefore

$$a^2bc \le \left(\frac{a(6-a)}{2}\right)^2.$$

Applying the inequality again, this time to a and $6-a$, gives

$$a(6-a) \le \left(\frac{a+(6-a)}{2}\right)^2 = 9,$$

therefore

$$\left(\frac{a(6-a)}{2}\right)^2 \le \left(\frac{9}{2}\right)^2.$$

Hence

$$a^2bc \le \left(\frac{a(6-a)}{2}\right)^2 \le \left(\frac{9}{2}\right)^2 = \frac{81}{4}.$$

Equality occurs when $b=c$ and $a=6-a$. Thus $a=3$ and $b=c=\frac{3}{2}$, so equality is attainable.

Therefore the maximum value is $\frac{81}{4}$.

Question 18

Find the total length of the intervals for which

$$(2\sin x - 1)\cos\left(2x - \frac{\pi}{6}\right)\tan x \le 0$$

for $0 < x < \pi$, with $x \neq \frac{\pi}{2}$.

A $\dfrac{5\pi}{6}$

B $\dfrac{\pi}{3}$

C $\dfrac{\pi}{4}$

D $\dfrac{5\pi}{12}$

E $\dfrac{\pi}{6}$

F $\dfrac{7\pi}{12}$

Solution 18

Answer: B

For this, we use the same method as for sketching a polynomial written as a product of linear factors. We do not need to sketch the curve itself; we only need to track how the sign changes at each root and discontinuity.

The first factor is zero when $2\sin x - 1 = 0$, giving $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

The second factor is zero when $\cos\left(2x - \frac{\pi}{6}\right) = 0$, giving $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$.

Also, $\tan x$ is undefined at $x = \frac{\pi}{2}$. We list $x = \frac{5\pi}{6}$ twice because two factors are zero there.

For x just greater than 0, the product is negative. The signs therefore alternate as follows:

x	0		$\frac{\pi}{6}$		$\frac{\pi}{3}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$		$\frac{5\pi}{6}$		π
sign		$-$	0	$+$	0	$-$	undefined	$+$	0	$-$	0	$+$	

There is no interval between the two appearances of $\frac{5\pi}{6}$; listing it twice simply records the two simultaneous sign changes.

Hence the inequality is satisfied on

$$\left(0, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$$

and at the isolated point $x = \frac{5\pi}{6}$. The isolated point has no length, so the total length is

$$\frac{\pi}{6} + \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{3}.$$

Question 19

Given that $x^3 + ax + 120 = 0$ has exactly two positive integer roots, what is the value of a?

The equation may or may not have other roots.

A -20

B -42

C -21

D -55

E -49

Solution 19

Answer: E

Let the two positive integer roots be r and s, where $r < s$.

Since both are roots,

$$r^3 + ar + 120 = 0$$

and

$$s^3 + as + 120 = 0.$$

Subtracting gives

$$r^3 - s^3 + a(r - s) = 0.$$

Since $r \neq s$, dividing by $r - s$ gives

$$r^2 + rs + s^2 + a = 0,$$

so

$$a = -(r^2 + rs + s^2).$$

Substituting this into $r^3 + ar + 120 = 0$ gives

$$r^3 - r(r^2 + rs + s^2) + 120 = 0,$$

which simplifies to

$$rs(r + s) = 120.$$

Since 120 is divisible by 5, at least one of r, s and $r+s$ must be divisible by 5. This leaves relatively few plausible pairs to check.

The factorisation $120 = 3 \cdot 5 \cdot 8$ suggests trying $r = 3$ and $s = 5$, since $r + s = 8$. Indeed,

$$3 \cdot 5 \cdot (3+5) = 120.$$

Therefore, the two positive integer roots are 3 and 5. Hence

$$a = -(3^2 + 3 \cdot 5 + 5^2) = -49.$$

Finally,

$$x^3 - 49x + 120 = (x-3)(x-5)(x+8),$$

so the equation has exactly two positive integer roots, 3 and 5.

Therefore, $a = -49$.

Question 20

You may use the following fact.

If f and g are strictly increasing functions defined for $x \geq 0$, and satisfy $f(g(x)) = x$ for all $x \geq 0$, then the graphs of f and g are reflections of each other in the line $y = x$.

Given that

$$\int_0^4 \left(2^{\sqrt{x}} - 1\right)\, dx = k\,,$$

find

$$\int_0^3 (\log_2(4-x))^2\, dx$$

in terms of k.

A $2k$

B 3

C $6 - k$

D $6 + k$

E $12 - k$

F k

G $9 - k$

Solution 20

Answer: E

First consider

$$f(x) = 2^{\sqrt{x}} - 1$$

and

$$g(x) = (\log_2(4-x))^2\,.$$

It is natural to check whether $f(g(x)) = x$, but this is not true.

However, reflecting the graph of $y = (\log_2(4-x))^2$ in the line $x = \frac{3}{2}$ does not change the area under the graph from $x = 0$ to $x = 3$. This reflection replaces x by $3 - x$, giving

$$g(x) = (\log_2(4 - (3 - x)))^2 = (\log_2(x + 1))^2 .$$

Now, for $x \geq 0$, we have $\log_2(x + 1) \geq 0$, so

$$f(g(x)) = 2^{\sqrt{(\log_2(x+1))^2}} - 1 = 2^{\log_2(x+1)} - 1 = x.$$

Therefore f and g are inverse functions, so their graphs are reflections of each other in the line $y = x$.

Since $f(0) = 0$ and $f(4) = 3$, the areas under the two graphs fill a rectangle of dimensions 4 by 3. Hence

$$\int_0^4 \left(2^{\sqrt{x}} - 1\right) dx + \int_0^3 (\log_2(x + 1))^2 \, dx = 12.$$

The first integral is k, so

$$\int_0^3 (\log_2(x + 1))^2 \, dx = 12 - k.$$

Finally, reflection in the line $x = \frac{3}{2}$ preserves the area, so

$$\int_0^3 (\log_2(4 - x))^2 \, dx = 12 - k.$$

Remark. Well done - you survived this horrific question, which combines two difficult geometric ideas: reflection in the line $y = x$ and reflection in a vertical line $x = a$. Each idea is tricky on its own; together, they make one hell of a puzzle!