JZ Mock Set E Paper 2 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

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Question 1

Let t be a real number, and define

$$f(x) = x^3 - 3tx + 1.$$

Consider the following statements.

P: f is increasing on the interval $[-1, 1]$.

Q: The equation $f(x) = 1$ has exactly one solution in the interval $[-1, 1]$.

A P is sufficient but not necessary for Q

B P is necessary but not sufficient for Q

C P is necessary and sufficient for Q

D P is neither sufficient nor necessary for Q

Solution 1

Answer: A

$$f'(x) = 3x^2 - 3t = 3(x^2 - t).$$

For f to be increasing on $[-1, 1]$, we need $x^2 - t \geq 0$ for every x in $[-1, 1]$, so $t \leq 0$.

Next, for Q:

$$f(x) = 1 \iff x^3 - 3tx = 0 \iff x(x^2 - 3t) = 0.$$

There is always the solution $x = 0$. Extra solutions occur exactly when $0 \leq t \leq \frac{1}{3}$. Therefore Q is true when $t < 0$, when $t = 0$, and when $t > \frac{1}{3}$; in other words, when $t \leq 0$ or $t > \frac{1}{3}$.

Thus P implies Q, but Q does not imply P, or the correct statement is: P is sufficient but not necessary for Q.

Question 2

Consider the following attempt to solve the equation $(x-2)\sqrt{x+3} = x^2 - 5x + 6$:

$$(x-2)\sqrt{x+3} = x^2 - 5x + 6 \quad \text{(I)}$$

$$(x-2)\sqrt{x+3} = (x-2)(x-3) \quad \text{(II)}$$

$$\sqrt{x+3} = x - 3 \quad \text{(III)}$$

$$x + 3 = (x-3)^2 \quad \text{(IV)}$$

$$x + 3 = x^2 - 6x + 9 \quad \text{(V)}$$

$$x^2 - 7x + 6 = 0 \quad \text{(VI)}$$

$$(x-1)(x-6) = 0 \quad \text{(VII)}$$

The solutions of the original equation are concluded to be $x = 1$ and $x = 6$.

Which one of the following is true?

A The conclusion is entirely correct.

B The conclusion is not entirely correct; the first error occurs in passing from (I) to (II).

C The conclusion is not entirely correct; the first error occurs in passing from (II) to (III).

D The conclusion is not entirely correct; the first error occurs in passing from (III) to (IV).

E The conclusion is not entirely correct; the first error occurs in passing from (IV) to (V).

F The conclusion is not entirely correct; the first error occurs in passing from (V) to (VI).

G The conclusion is not entirely correct; the first error occurs in passing from (VI) to (VII).

Solution 2

Answer: C

The factorisation in passing from (I) to (II) is correct, since $x^2 - 5x + 6 = (x-2)(x-3)$.

The first error occurs in passing from (II) to (III). Dividing both sides by $x - 2$ assumes that $x \neq 2$, but $x = 2$ satisfies the original equation, since both sides are 0. This solution is therefore lost.

For $x \neq 2$, equation (III) is $\sqrt{x+3} = x - 3$. Since the left-hand side is non-negative, any solution must satisfy $x \geq 3$. Squaring gives $(x-1)(x-6) = 0$, so the only possible value satisfying $x \geq 3$ is $x = 6$.

Thus the solutions of the original equation are $x = 2$ and $x = 6$. The stated conclusion $x = 1$ and $x = 6$ is not entirely correct, and the first error occurs in passing from (II) to (III).

Question 3

A real number x is such that **exactly one** of the following six statements is true.

I $x^2 < 9$

II $|x - 1| < 1$

III $x, 2, 4$ are the first three terms of a geometric sequence with positive common ratio

IV $\log_2(x + 1)$ is defined and less than 1

V x is a positive solution of $u^2 - 3u + 2 = 0$

VI the circle $X^2 + Y^2 = 1$ meets the line $X = x$ in two distinct points

A I

B II

C III

D IV

E V

F VI

Solution 3

Answer: A

It turns out each of II to VI implies I!

Indeed, II gives $0 < x < 2$, III gives $x = 1$, IV gives $-1 < x < 1$, V gives $x = 1$ or $x = 2$, and VI gives $-1 < x < 1$. In each case, $x^2 < 9$.

Therefore, if any of II to VI were true, then I would also be true. Since exactly one statement is true, the only possible true statement is I.

Note it is also possible for I to be the only true statement, for example when $x = \frac{5}{2}$.

Question 4

Find the sum of all solutions x in the interval $0 \le x < 2\pi$ to

$$|\sin x| + |\cos x| = 1.$$

A π

B $\dfrac{3\pi}{2}$

C 2π

D $\dfrac{5\pi}{2}$

E 3π

F $\dfrac{7\pi}{2}$

Solution 4

Answer: E

$$a = |\sin x|, \qquad b = |\cos x|.$$

Then $a \ge 0$, $b \ge 0$, and

$$a^2 + b^2 = 1.$$

We also need $a + b = 1$. Squaring gives

$$a^2 + b^2 + 2ab = 1,$$

so $2ab = 0$. Thus either $|\sin x| = 0$ or $|\cos x| = 0$.

In $0 \le x < 2\pi$, the solutions are

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

Their sum is 3π.

Remark: the answer can also be **seen** from a sketch of $|\sin x|$ and $|\cos x|$ on the same plot.

Question 5

The real numbers x and y are such that exactly two of the following five statements are true. Which two are true?

I $\quad x < y$

II $\quad x < |y|$

III $\quad |x| < |y|$

IV $\quad x \geq 0$

V $\quad x < 0$

A I and IV

B I and V

C II and IV

D II and V

E III and IV

F III and V

Solution 5

Answer: D

Statements IV and V are opposites, so exactly one of them is true. Therefore exactly one of I, II and III is true.

I cannot be the only true one: if $x < y$, then $x < y \leq |y|$, so II would also be true.

III cannot be the only true one: if II is false, then $x \geq |y|$, and combining this with III gives $x \geq |y| > |x| \geq x$, which is impossible.

So II is the only true statement among the first three, and both I and III are false. From II and the falsity of III, we get $x < |y| \leq |x|$. If x were non-negative, then $|x| = x$, giving $x < x$, which is impossible. Hence $x < 0$, so V is true and IV is false.

Therefore the true statements are II and V. For example, $x = y = -1$ satisfies these conditions.

Question 6

Let $c > 0$ be a fixed real constant. For each real number b, define

$$f_b(x) = x^2 - 2bx + c,$$

and let P_b denote the vertex of the graph $y = f_b(x)$. As b varies over the real numbers, the point P_b traces out a curve Γ in the (x, y)-plane.

Consider the following three statements.

(I) The curve Γ has equation $y = c + x^2$.

(II) For all real numbers $b_1 < b_2$, the point P_{b_2} lies strictly to the right of, and strictly below, the point P_{b_1}.

(III) There is a unique real value of b for which f_b has a repeated root.

Which of the statements are necessarily true?

A (I) only

B (II) only

C (III) only

D (I) and (II) only

E (I) and (III) only

F (II) and (III) only

G all of them

H none of them

Solution 6

Answer: H

Completing the square, $f_b(x) = (x - b)^2 + (c - b^2)$, so the vertex is $P_b = (b,\, c - b^2)$.

(I) Setting $X = b$ and $Y = c - b^2$, the locus satisfies $Y = c - X^2$. As b ranges over $\mathbb{R}$, X also ranges over $\mathbb{R}$, so Γ is exactly the parabola $y = c - x^2$. Therefore (I) is false.

(II) Recall that the actual curve of P_b is $y = c - x^2$, this is an inverted quadratic, and so (II) is not true for all the points to the left of the maximum of the curve. Therefore (II) is false.

(III) Repeated root occurs if and only if $(c - b^2) = 0$, and since $c > 0$, there are two values of b that works, $\pm\sqrt{c}$, hence (III) is also false, as (III) require there to be always precisely one value of b for which f_b has repeated roots.

So none of them are true.

Question 7

Positive real numbers x, y and z are such that

$$x, y, z$$

are consecutive terms of an arithmetic sequence, while

$$\log_2 x, \log_2 y, \log_2 z$$

are consecutive terms of an arithmetic sequence.

It is also known that

$$x + y + z = 12.$$

Find xyz.

A 32

B 48

C 60

D 64

E 72

F 96

Solution 7

Answer: D

Since x, y, z are consecutive terms of an arithmetic sequence,

$$x + z = 2y.$$

Since $\log_2 x, \log_2 y, \log_2 z$ are consecutive terms of an arithmetic sequence,

$$\log_2 x + \log_2 z = 2\log_2 y.$$

This means

$$xz = y^2.$$

So x, y, z are also consecutive terms of a geometric sequence.

A positive sequence which is **both** arithmetic and geometric must have all three terms equal. Therefore $x = y = z = 4$, and so $xyz = 64$.

Question 8

Consider triangles ABC such that $AB = 1$, $\sin A = \frac{1}{3}$ and $\cos C = x$.

What is the greatest possible value of x for which there is **exactly one** distinct triangle satisfying these conditions?

A

$$\frac{1}{3}$$

B

$$\frac{\sqrt{2}}{3}$$

C

$$\frac{2}{3}$$

D

$$\frac{2\sqrt{2}}{3}$$

E

$$\frac{1+\sqrt{2}}{3}$$

Solution 8

Answer: D

Here is the intuition for this question.

For there to be two distinct triangles, $\sin A = \frac{1}{3}$ is ambiguous in A, since $\sin A = \sin(180^\circ - A)$. To find the answer to this question quickly, it is useful to think of A as approximately 20°. Then A could also be 160°. Therefore, for there to be two distinct triangles, C can be upto, not include 20°; otherwise, two of the angles would already exceed 180°. Hence, the smallest value of C for which there is only one triangle is 20°. Since $\cos C$ is decreasing between 0° and 180°, the greatest value of $\cos C = x$ therefore occurs when $C = 20^\circ$, or whatever the actual value of A is. In other words, the answer is $\cos A$, of the acute version of A.

Now let's formally prove it.

Let $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$. Then the possible values of A are α and $180^\circ - \alpha$.

For a fixed $x = \cos C$, there is exactly one value of C in $0^\circ < C < 180^\circ$.

If $A = \alpha$, a triangle exists when $C < 180^\circ - \alpha$. If $A = 180^\circ - \alpha$, a triangle exists when $C < \alpha$.

Hence there is exactly one triangle when $\alpha \leq C < 180^\circ - \alpha$. Since $\cos C$ decreases as C increases, the greatest value of x occurs when $C = \alpha$.

Therefore,

$$x = \cos\alpha = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}.$$

Question 9

For real t, define

$$A(t) = \int_{-1}^{1} |x^2 - t| \, dx.$$

Find the minimum possible value of $A(t)$.

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

F 1

Solution 9

Answer: C

If $t < 0$ or $t > 1$, moving t towards the interval $[0, 1]$ decreases the integral, so the minimum occurs with $0 \le t \le 1$.

For $0 \le t \le 1$, the graphs $y = x^2$ and $y = t$ intersect at $x = \pm\sqrt{t}$. By symmetry,

$$A(t) = 2\left(\int_{0}^{\sqrt{t}} (t - x^2) \, dx + \int_{\sqrt{t}}^{1} (x^2 - t) \, dx\right).$$

Evaluating the integrals gives

$$A(t) = \frac{2}{3} - 2t + \frac{8}{3}t^{3/2}.$$

Therefore,

$$A'(t) = -2 + 4\sqrt{t}.$$

Setting $A'(t) = 0$ gives $\sqrt{t} = \frac{1}{2}$, so $t = \frac{1}{4}$. Since $A'(t) < 0$ before this value and $A'(t) > 0$ afterwards, this gives the minimum.

Finally find A at $t = \frac{1}{4}$,

$$A\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Question 10

A polynomial $p(x)$ has degree at most 3. When $p(x)$ is divided by $x-1$, $x-2$ and $x-4$, the remainders are 1, 2 and 4 respectively.

It is also known that $p(0)=0$.

Find the coefficient of x^3 in $p(x)$.

A -3

B -2

C -1

D 0

E 1

F 2

G 3

Solution 10

Answer: D

The remainder conditions imply:

$$p(1)=1, \qquad p(2)=2, \qquad p(4)=4,$$

and we are also given $p(0)=0$.

Now consider

$$q(x)=p(x)-x.$$

The polynomial q has degree at most 3, and it has roots

$$0,1,2,4.$$

A non-zero polynomial of degree at most 3 cannot have four distinct roots, so $q(x)$ is the zero polynomial. Hence

$$p(x)=x.$$

The coefficient of x^3 is 0.

Question 11

Let f be a real function and let a be a real number.

We say that f is **smoothly joined** at $x = a$ if and only if, for every real number $h > 0$, there exists a real number $d > 0$ such that, for every real number x, if $|x - a| < d$, then $|f(x) - f(a)| < h$.

Which of the following means that f is **not smoothly joined** at $x = a$?

A For every real number $h > 0$, for every real number $d > 0$, there exists a real number x such that $|x - a| < d$ and $|f(x) - f(a)| \geq h$.

B There exists a real number $h > 0$ such that there exists a real number $d > 0$ such that, for every real number x, if $|x - a| < d$, then $|f(x) - f(a)| \geq h$.

C There exists a real number $h > 0$ such that, for every real number $d > 0$, there exists a real number x such that $|x - a| < d$ and $|f(x) - f(a)| \geq h$.

D There exists a real number $h > 0$ such that, for every real number $d > 0$, for every real number x, if $|x - a| < d$, then $|f(x) - f(a)| \geq h$.

E For every real number $h > 0$, there exists a real number $d > 0$ such that, for every real number x, if $|x - a| < d$, then $|f(x) - f(a)| \geq h$.

Solution 11

Answer: C

The definition says that f is **smoothly joined** at $x = a$ exactly when

$$\text{for every } h > 0, \text{ there exists } d > 0, \text{ such that for every } x, \text{ if } |x - a| < d, \text{ then } |f(x) - f(a)| < h.$$

To negate this, reverse each quantifier and negate the final implication, starting from the outside and working inwards, and recursively do this until the innermost statement is reached.

So start with: there exists some $h > 0$ for which the following statement is false.

The false version of the following statement is its negation. Therefore we need:

for all $d > 0$, the following statement is false.

Now the false version of the following statement is also its negation. Therefore we get:

there must exist some x with the desired property $|x - a| < d$, for which the following statement is false, that is, $|f(x) - f(a)| \geq h$.

Putting it all together, f is therefore **not smoothly joined** at $x = a$ means:

there exists $h > 0$ such that for every $d > 0,$ there exists x such that $|x - a| < d$ and $|f(x) - f(a)| \geq h$.

This is the third statement.

Question 12

Let f be a function. Each of the following is a statement about all real numbers a (for instance, the first reads: for every a, if $f(x) > 0$ for all $x > a$, then $f(a) < 0$). **Exactly one** of them is true. Which one is it?

A $f(x) > 0$ for all $x > a$ only if $f(a) < 0$

B $f(a) < 0$ and $f(x) > 0$ for all $x > a$ cannot both hold

C $f(x) > 0$ for all $x > a$ is necessary for $f(a) < 0$

D if $f(a) \geq 0$ then $f(x) \leq 0$ for some $x > a$

E $f(x) > 0$ for all $x > a$ implies $f(a) \geq 0$

F $f(a) \geq 0$ if and only if $f(x) \leq 0$ for some $x > a$

G $f(a) < 0$ is sufficient for there to exist $x > a$ with $f(x) \leq 0$

Solution 12

Answer: C

Write P for "$f(x) > 0$ for all $x > a$" and Q for "$f(a) < 0$". The statements become:

A. $P \Rightarrow Q$

B. $Q \Rightarrow \neg P$

C. $Q \Rightarrow P$

D. $\neg Q \Rightarrow \neg P$, the contrapositive of $P \Rightarrow Q$

E. $P \Rightarrow \neg Q$, the contrapositive of $Q \Rightarrow \neg P$

F. $\neg Q \Leftrightarrow \neg P$, which is equivalent to $P \Leftrightarrow Q$

G. $Q \Rightarrow \neg P$

Therefore, $P \Rightarrow Q$ appears twice, in A and D, while $Q \Rightarrow \neg P$ appears three times, in B, E and G.

Statement F implies $P \Rightarrow Q$, so F cannot be the only true statement.

This leaves only C, which is $Q \Rightarrow P$, given exactly one is true, must be C.

Question 13

For some positive integer n, three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio $1:3:5$. What is the value of n?

A 5

B 7

C 12

D 15

E 18

Solution 13

Answer: B

Suppose the three consecutive coefficients are

$$\binom{n}{r}, \quad \binom{n}{r+1}, \quad \binom{n}{r+2}.$$

Since their ratio is $1:3:5$,

$$\frac{\binom{n}{r+1}}{\binom{n}{r}} = \frac{n-r}{r+1} = 3,$$

so $n = 4r+3$.

Also,

$$\frac{\binom{n}{r+2}}{\binom{n}{r+1}} = \frac{n-r-1}{r+2} = \frac{5}{3}.$$

Substituting $n = 4r+3$ gives

$$\frac{3r+2}{r+2} = \frac{5}{3},$$

so $9r+6 = 5r+10$, giving $r=1$. Therefore

$$n = 4(1)+3 = 7.$$

The answer is 7.

Question 14

The two diagonals of the convex quadrilateral Q have equal length. A quadrilateral is convex if each of its interior angles is less than $180°$.

Consider the following statements.

(I) The four midpoints of the sides of Q are the vertices of a rhombus.

(II) At least one pair of opposite sides of Q is parallel.

(III) Q has at least one line of symmetry.

(IV) Let A, B, C, D be the vertices of Q, labelled in order, then $AB^2 + CD^2 = BC^2 + DA^2$.

Which of these statements is/are **necessarily** true for the quadrilateral Q?

A (I) only

B (I) and (II) only

C (I) and (III) only

D (II) and (IV) only

E (II) and (III) only

F (I) and (IV) only

G (I), (III) and (IV) only

H (II), (III) and (IV) only

I all of them

J none of them

Solution 14

Answer: A

Let A, B, C, D be the vertices of Q in order, and let E, F, G, H be the midpoints of AB, BC, CD, DA, respectively.

From the definitions of the midpoints,

$$\overrightarrow{EF} = \frac{1}{2}\overrightarrow{AC}, \qquad \overrightarrow{FG} = \frac{1}{2}\overrightarrow{BD},$$

and similarly,

$$\overrightarrow{GH} = -\frac{1}{2}\overrightarrow{AC}, \qquad \overrightarrow{HE} = -\frac{1}{2}\overrightarrow{BD}.$$

Thus opposite sides of $EFGH$ are parallel and equal, so $EFGH$ is a parallelogram. Also,

$$EF = \frac{1}{2}AC, \qquad FG = \frac{1}{2}BD.$$

Since $AC = BD$, we have $EF = FG$. Therefore the parallelogram $EFGH$ has equal adjacent sides, so it is a rhombus. Hence statement (I) is necessarily true.

Now for statements (II) and (III), consider possible construction of Q by drawing the two diagonals first, it becomes immediately obvious that both are not generally necessarily true, as counterexamples can be easily found with this construction.

(IV) feels not necessarily true, but is best confirmed with a counterexample. Take $A = (0, 0)$, $B = (1, -1)$, $C = (5, 0)$ and $D = (4, 3)$, listed in order. These points form a convex quadrilateral, and the diagonals have equal length, since $AC = 5$ and $BD = 5$.

However, $AB^2 + CD^2 = 2 + 10 = 12$, while $BC^2 + DA^2 = 17 + 25 = 42$. Therefore $AB^2 + CD^2 \neq BC^2 + DA^2$, and (IV) is not necessarily true.

So only (I) is necessarily true.

Question 15

For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. The fractional part of x is then defined by

$$\{x\} = x - \lfloor x \rfloor,$$

so $0 \leq \{x\} < 1$.

For example, $\lfloor 3.7 \rfloor = 3$, so $\{3.7\} = 3.7 - 3 = 0.7$.

Also, $\lfloor -1.4 \rfloor = -2$, since $-2 \leq -1.4 < -1$. Therefore $\{-1.4\} = -1.4 - (-2) = 0.6$.

Define

$$f(x) = \{\{2x\} + \{3x\}\}.$$

How many values of x with $0 \leq x < 1$ satisfy $f(f(x)) = 1 - x$?

A 20

B 21

C 24

D 25

E 26

F 30

G 31

Solution 15

Answer: D

For any real t, Write $2t = \lfloor 2t \rfloor + \{2t\}$ and $3t = \lfloor 3t \rfloor + \{3t\}$. Then

$$5t = \lfloor 2t \rfloor + \lfloor 3t \rfloor + \{2t\} + \{3t\}.$$

Since $\lfloor 2t \rfloor + \lfloor 3t \rfloor$ is an integer, adding it does not change the fractional part. Hence

$$\{5t\} = \{\{2t\} + \{3t\}\}.$$

For any real t,

$$f(t) = \{\{2t\} + \{3t\}\} = \{5t\}.$$

Hence

$$f(f(x)) = \{5\{5x\}\} = \{25x\}.$$

Suppose $\{25x\} = 1 - x$. Writing $n = \lfloor 25x \rfloor$ gives $25x - n = 1 - x$, so

$$x = \frac{n+1}{26}.$$

This gives the 25 values

$$x = \frac{1}{26}, \frac{2}{26}, \dots, \frac{25}{26}.$$

Question 16

Let f be a positive function defined on $[0, \infty)$ and twice differentiable on $(0, \infty)$. Define

$$I = \int_0^4 f(x)\, dx,$$

and let J_n be the trapezium rule estimate of I using n strips of equal width, where n is a positive integer.

Which of the following statements are necessarily true?

(1) If $f''(x) > 0$ for all $0 < x < 4$, then $J_n > I$ for every positive integer n.

(2) If $f''(x) \geq 0$ and $f'(x) > 0$ for all $0 < x < 4$, then $J_{n+1} < J_n$ for every positive integer n.

(3) If $I < J_n$ for some positive integer n, then

$$\int_0^4 f(4 - x)\, dx < J_{n+1}.$$

A (1) only.

B (2) only.

C (3) only.

D (1) and (2) only.

E (1) and (3) only.

F (2) and (3) only.

G none of them.

H all of them

Solution 16

Answer: A

Statement (1) is true. Since $f''(x) > 0$ on $(0, 4)$, the graph of f is strictly convex. On each strip, the chord joining the endpoints of the graph lies strictly above the graph inside the strip. Therefore, the area of each trapezium is greater than the corresponding area under the curve. Adding the areas of all the strips gives $J_n > I$.

For statement (2), $f(x) = x + 1$ is a counterexample, since $f(x) > 0$ on $[0, \infty)$, while $f''(x) = 0$ and $f'(x) = 1 > 0$. Since the trapezium rule is exact for a linear function,

$$J_{n+1} = J_n = I$$

for every positive integer n. Therefore, statement (2) is false.

For statement (3), reflection does not change the value of the integral:

$$\int_0^4 f(4-x)\,dx = \int_0^4 f(x)\,dx = I.$$

Therefore, statement (3) is really claiming that

$$I < J_n \quad \Longrightarrow \quad I < J_{n+1}.$$

However, without any condition on the curvature of f, this is not necessarily true. Increasing the number of strips can change an overestimate into an underestimate. A suitable polynomial counterexample can be constructed for which $J_n > I$ but $J_{n+1} < I$. Therefore, statement (3) is false.

Therefore, only statement (1) is true.

Remark: Just to be rigorous,for a counterexample of (3), for example, take

$$f(x) = 2 + 2x - \frac{(x(4-x))^3}{64}.$$

This curve passes through $(0, 2)$ and $(4, 10)$, but dips below the chord joining these points, particularly near $x = 2$. Thus the one-strip estimate is too large, while the extra midpoint used by the two-strip estimate makes it too small.

Indeed,

$$J_1 = 24, \qquad I = 22.17, \qquad J_2 = 22,$$

so

$$J_1 > I > J_2.$$

Question 17

A sphere has volume 1 cubic unit. A regular tetrahedron is inscribed in the sphere, so that all four vertices of the tetrahedron lie on the sphere. A second sphere is then inscribed in this tetrahedron, so that it is tangent to all four faces of the tetrahedron.

This process is repeated indefinitely: inside each sphere, a regular tetrahedron is inscribed, and inside that tetrahedron another sphere is inscribed.

Find the sum of the volumes of all the spheres formed, including the original sphere.

A

$$\frac{5+\sqrt{5}}{4}$$

B

$$\frac{5}{4}$$

C

$$\frac{27}{26}$$

D

$$\frac{3+\sqrt{3}}{2}$$

E

$$\frac{9+3\sqrt{2}}{7}$$

F

$$\frac{4}{3}$$

Solution 17

Answer: C

The volumes of the successive spheres clearly form a geometric sequence. Therefore, it is enough to determine the ratio between the radius of a sphere inscribed in a regular tetrahedron and the radius of a sphere circumscribed about that tetrahedron. Once this ratio is known, the ratio of the volumes can be deduced immediately.

Let $ABCD$ be the regular tetrahedron, with ABC as its base, and let O be the centre of the equilateral triangle ABC. Since we only care about ratios of lengths, we may choose the side length of the tetrahedron. Let us choose it to be 2, which is particularly convenient.

The height of the equilateral triangle ABC is

$$\frac{\sqrt{3}}{2} \times 2 = \sqrt{3}.$$

The centre O lies two-thirds of the way from a vertex to the opposite side, so

$$AO = \frac{2}{3} \times \sqrt{3} = \frac{2\sqrt{3}}{3}.$$

Since DO is perpendicular to the base, triangle AOD is right-angled. Therefore,

$$DO^2 = AD^2 - AO^2 = 2^2 - \left(\frac{2\sqrt{3}}{3}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3},$$

so

$$DO = \frac{4}{\sqrt{6}}.$$

Let P be the centre of the tetrahedron, and let $OP = t$. By symmetry, P lies on DO. Since P is equally distant from all four vertices,

$$PA = PD.$$

Now

$$PA^2 = AO^2 + OP^2 = \frac{4}{3} + t^2$$

and

$$PD = DO - OP = \frac{4}{\sqrt{6}} - t.$$

Therefore,

$$\left(\frac{4}{\sqrt{6}} - t\right)^2 = \frac{4}{3} + t^2.$$

Expanding gives

$$\frac{8}{3} - \frac{8}{\sqrt{6}}t + t^2 = \frac{4}{3} + t^2,$$

so

$$t = \frac{1}{\sqrt{6}}.$$

Hence the distance from the centre to each vertex is

$$PD = \frac{4}{\sqrt{6}} - \frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}}.$$

Thus, the radius of the inscribed sphere is $\frac{1}{\sqrt{6}}$, while the radius of the circumscribed sphere is $\frac{3}{\sqrt{6}}$. Their radii are therefore in the ratio $1 : 3$.

Hence their volumes are in the ratio $1 : 27$. The successive sphere volumes therefore form a convergent geometric sequence with first term 1 and common ratio $\frac{1}{27}$. Their sum is

$$\frac{1}{1-\frac{1}{27}} = \frac{27}{26}.$$

Question 18

In a geometric progression, the third term is $3-\sqrt{5}$ and the seventh term is $18-8\sqrt{5}$. Given that every term of the progression is positive, what is the sum to infinity?

A $\sqrt{5}-1$

B 2

C $\sqrt{5}+1$

D $3-\sqrt{5}$

E $2\sqrt{5}+4$

F $\dfrac{3+\sqrt{5}}{2}$

G $3+\sqrt{5}$

Solution 18

Answer: G

Let the first term be a and common ratio r. From $ar^2 = 3-\sqrt{5}$ and $ar^6 = 18-8\sqrt{5}$, divide to get

$$r^4 = \frac{18-8\sqrt{5}}{3-\sqrt{5}}.$$

Rationalise by multiplying numerator and denominator by $(3+\sqrt{5})$: numerator $(18-8\sqrt{5})(3+\sqrt{5}) = 54+18\sqrt{5}-24\sqrt{5}-40 = 14-6\sqrt{5}$; denominator $9-5=4$.

Therefore

$$r^4 = \frac{7-3\sqrt{5}}{2} = \frac{14-6\sqrt{5}}{4} = \frac{5+9-2\cdot 3\cdot\sqrt{5}}{4} = \left(\frac{3-\sqrt{5}}{2}\right)^2.$$

Hence $r^2 = \frac{3-\sqrt{5}}{2}$, taking the non-negative square root since $r^2 \geq 0$. Then $r = \pm\frac{\sqrt{5}-1}{2}$ since $\left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$. The positivity condition forces $r>0$, so $r = \frac{\sqrt{5}-1}{2}$.

Now $a = \frac{3-\sqrt{5}}{r^2} = \frac{3-\sqrt{5}}{(3-\sqrt{5})/2} = 2$. Finally $1-r = \frac{3-\sqrt{5}}{2}$, so

$$S_\infty = \frac{2}{(3-\sqrt{5})/2} = \frac{4}{3-\sqrt{5}} = \frac{4(3+\sqrt{5})}{9-5} = \frac{4(3+\sqrt{5})}{4} = 3+\sqrt{5}.$$

Question 19

In a previous JZ Maths mock paper, candidates were introduced to the following two question-setting policies.

Policy 1. A multiple-choice question involving five statements must be made as difficult as possible: a student must determine the truth value of all five statements in order to guarantee identifying the correct option. In particular, checking any four statements must not be sufficient.

Policy 2. Subject to Policy 1, the question must contain as few answer options as possible, so that it appears easier than it really is.

The disclaimer accompanying that question claimed that no such policies officially existed.

Alan Turing, a student sitting the next JZ Maths mock paper, has secretly learnt that JZ Maths does, in fact, follow both policies religiously. JZ Maths is, of course, completely unaware of this serious breach of question-setting security!

Alan is presented with a question containing five statements, labelled $(1), (2), (3), (4), (5)$. He does not know how many of the statements are true. The answer options are shown below, with each option specifying exactly which statements are true.

Without bothering to check the truth of any statement, Alan is able to identify the correct option.

Which option does he choose?

A Statements $(1), (3), (5)$ only.

B Statements $(3), (5)$ only.

C Statements $(1), (2), (3), (5)$ only.

D Statements $(1), (5)$ only.

E Statements $(1), (3), (4), (5)$ only.

F Statements $(1), (3)$ only.

Solution 19

Answer: A

By Policy 1, checking all statements except for statement i must not be enough to deduce the correct option. Once four statements have been checked, there are only two possible truth patterns left, according to whether statement i is true or false. Therefore, both possibilities must appear among the options.

The two options left must agree on all statements except statement i, as this is the only way that, after checking all statements except statement i, we are still unable to distinguish between the last two options.

Since this logic holds for every statement i, we can use it to check whether an option could be the correct option.

For example, suppose the correct option were $\{3, 5\}$.

Checking all but statement (1) should leave $\{3, 5\}$ and $\{1, 3, 5\}$ undetermined. Both are there, so this is consistent.

Checking all but statement (2) should leave $\{3, 5\}$ and $\{2, 3, 5\}$ undetermined. However, $\{2, 3, 5\}$ is not one of the options, so this is not consistent. Therefore, $\{3, 5\}$ is not the correct option.

Now, let us check whether $\{1, 3, 5\}$ is the correct option.

Checking all but statement (1) should leave $\{1, 3, 5\}$ and $\{3, 5\}$ undetermined. Both are there, so this is consistent.

Checking all but statement (2) should leave $\{1, 3, 5\}$ and $\{1, 2, 3, 5\}$ undetermined. Both are there, so this is consistent.

Checking all but statement (3) should leave $\{1, 3, 5\}$ and $\{1, 5\}$ undetermined. Both are there, so this is consistent.

Checking all but statement (4) should leave $\{1, 3, 5\}$ and $\{1, 3, 4, 5\}$ undetermined. Both are there, so this is consistent.

Checking all but statement (5) should leave $\{1, 3, 5\}$ and $\{1, 3\}$ undetermined. Both are there, so this is consistent.

This is consistent in all cases. Therefore, $\{1, 3, 5\}$ is the correct option, and Alan deftly selects the first option without checking any of the five statements, thereby defeating the purpose of both policies using only his knowledge of the policies themselves. The policies were designed to maximise the difficulty of the question, but ironically made the five statements completely irrelevant!

Remark: Policy 2 is actually redundant in this question. Whatever the correct truth pattern is, Policy 1 requires the correct option together with five further options, obtained by changing the truth value of exactly one of the five statements. These six options are always sufficient, regardless of how many of the statements are true, which is a fact that Alan, of course, had already deduced. Therefore, Policy 1 requires at least six options, and six options are enough. Since Alan can see that the question contains exactly six options, Policy 2 gives him no additional information in this case.

Question 20

Warning: Students should be aware that JZ Maths policies may change at any time, entirely at the whim of the unstable chief mock examiner - Mr JZ.

Statement P: ABC is a right-angled triangle.

Which of the following statements are true?

(1) Statement P is true if and only if triangle ABC can be divided by a line segment into two smaller triangles, each of which is similar to triangle ABC.

(2) Statement P is necessary for triangle ABC to be divisible by a line segment into two smaller triangles whose areas, together with the area of triangle ABC, can be arranged to form a geometric sequence with common ratio not equal to 1.

(3) P is a necessary condition, that must be satisfied by all the triangles made when a convex kite is divided into four triangles by its diagonals.

(4) Statement P is sufficient for triangle ABC to be determined uniquely up to congruence by the length of its hypotenuse and the length of the median to its hypotenuse.

The median is the line segment joining the opposite vertex to the midpoint of the hypotenuse.

(5) Statement P is true if and only if four congruent copies of triangle ABC can be assembled edge-to-edge, without gaps or overlaps, so that their union is exactly a rhombus.

A Statements $(1), (3), (5)$ only.

B Statements $(3), (5)$ only.

C Statements $(1), (2), (3), (5)$ only.

D Statements $(1), (5)$ only.

E Statements $(1), (3), (4), (5)$ only.

F Statements $(1), (3)$ only.

Solution 20

Answer: F

Remark 1: Did you gamble? You would have lost!

For statement (1), suppose first that ABC is right-angled. Drawing the altitude from the right-angled vertex to the hypotenuse divides ABC into two smaller triangles, each similar to ABC.

Conversely, suppose a line segment divides ABC into two smaller triangles, each similar to ABC. The line segment must join a vertex to a point on the opposite side. The two angles formed at this point are supplementary.

Since both smaller triangles are similar to ABC, these two supplementary angles must both occur among the angles of ABC. They cannot be two distinct angles of ABC, since the sum of any two angles of a triangle is less than $180°$. Therefore they are equal, so each is $90°$. Hence ABC is right-angled.

Thus statement (1) is true.

For statement (2), let the area of ABC be S, and let $r > 1$ satisfy

$$r^2 = r + 1.$$

A line segment from a vertex to the opposite side can divide any triangle in any prescribed positive area ratio. In particular, it can produce two triangles with areas

$$\frac{S}{r^2} \quad \text{and} \quad \frac{S}{r}.$$

Indeed,

$$\frac{S}{r^2} + \frac{S}{r} = S,$$

since $r^2 = r + 1$.

The three areas can therefore be arranged as

$$\frac{S}{r^2}, \quad \frac{S}{r}, \quad S,$$

which form a geometric sequence with common ratio $r \neq 1$.

This construction works for any triangle, so being right-angled is not necessary. Thus statement (2) is false.

For statement (3), the diagonals of a convex kite are perpendicular. Therefore each of the four triangles formed by the diagonals has a right angle at the point where the diagonals intersect.

Thus statement (3) is true.

Statement (4) is false! In any right-angled triangle, the midpoint of the hypotenuse is equidistant from all three vertices. To see this, draw through the midpoint of the hypotenuse two lines parallel to the two perpendicular sides. Together with the median, these divide the original triangle into four congruent triangles.

Therefore, the median to the hypotenuse always has length equal to half the length of the hypotenuse. Hence, knowing the length of the hypotenuse already determines the length of the median, so the median provides no additional information. The length of the hypotenuse alone clearly does not uniquely determine a right-angled triangle up to congruence.

For statement (5), consider a rhombus with side length 2 and acute angle 60°. Join the midpoints of a pair of opposite sides. This divides the rhombus into two congruent parallelograms, each with adjacent side lengths 1 and 2.

Divide each parallelogram along its longer diagonal. This produces four congruent triangles which fill the rhombus edge-to-edge, without gaps or overlaps.

In each triangle, the sides of lengths 1 and 2 enclose an angle of 120°. If the third side has length d, then

$$d^2 = 1^2 + 2^2 - 2(1)(2)\cos 120^\circ = 1 + 4 + 2 = 7.$$

The triangle is not right-angled, since

$$1^2 + 2^2 = 5 \neq 7.$$

Therefore four congruent copies of a non-right-angled triangle can be assembled to form a rhombus.

Thus statement (5) is false.

Therefore the true statements are

statements (1) and (3) only.

Remark 2: Well done, you survived these mock paper sets, best of luck!