TMUA

COMMUNITY PAPERS

# Cryptokyo TMUA Problem Set

**Recommended time:** 75 minutes

**Calculators:** not permitted

**Format:** 19 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

**Paper guidance**

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

**Source credit:** Cryptokyo

## Question 1

Evaluate the integral

$$\int_0^1 \frac{1-x}{1-\sqrt{x}}\,\mathrm{d}x$$

**A** $\frac{2}{3}$

**B** $1$

**C** $\frac{4}{3}$

**D** $\frac{3}{2}$

**E** $\frac{5}{3}$

**F** $2$

## Question 2

Rationalise the denominator of

$$\frac{1}{1+\sqrt{3}+\sqrt{5}}$$

**A** $\dfrac{7+3\sqrt{3}-\sqrt{5}-2\sqrt{15}}{11}$

**B** $\dfrac{7-3\sqrt{3}+\sqrt{5}-2\sqrt{15}}{11}$

**C** $\dfrac{7+3\sqrt{3}-\sqrt{5}+2\sqrt{15}}{11}$

**D** $\dfrac{1-\sqrt{3}-\sqrt{5}}{11}$

**E** $\dfrac{7+3\sqrt{3}-\sqrt{5}-2\sqrt{15}}{9}$

## Question 3

Find the coefficient of $x^6$ in the expansion of

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)$$

**A** 2

**B** 3

**C** 4

**D** 5

**E** 6

**F** 7

## Question 4

A quadratic polynomial f($x$) has two distinct real roots $\alpha$ and $\beta$. We are given that $\alpha\beta = 1$ and that $\alpha + \beta = \gamma$.

What is the possible range of values of $\gamma$?

**A** $|\gamma| < 2$

**B** $\gamma > 2$

**C** $|\gamma| > 4$

**D** $|\gamma| \geq 2$

**E** $|\gamma| > 2$

## Question 5

The polynomial $x^2 + bx + c$ has roots $\alpha$ and $\beta$.

Find $\alpha^4 + \beta^4$ in terms of $b$ and $c$.

**A** $b^4 - 4b^2c + 4c^2$

**B** $b^4 - 2b^2c + 2c^2$

**C** $b^4 + 4b^2c + 2c^2$

**D** $b^4 - 4b^2c - 2c^2$

**E** $b^4 - 4b^2c + 2c^2$

## Question 6

Find $\mathrm{f}'(2)$ for

$$\mathrm{f}(x) = \frac{x^3 + 8x^2 + 11x - 20}{x^2 + 4x - 5}$$

**A** $-1$

**B** $\frac{1}{9}$

**C** $1$

**D** $2$

**E** $3$

## Question 7

Without using a calculator, compute the exact value of $\sqrt[3]{19.683}$.

**A** 1.9

**B** 2.7

**C** 3.3

**D** 9.0

**E** 27.0

## Question 8

How many positive integers are there less than 1000 that are **not** divisible by 2 or 3?

**A** 166

**B** 333

**C** 334

**D** 499

**E** 666

**F** 667

## Question 9

For which values of $a$ does the equation

$$3^{2x} - a \times 3^{x+1} + a^2 = 0$$

have two distinct real solutions?

I. $a < 0$

II. $a > 0$

III. No such values of $a$

IV. $a = 0$

**A** I only

**B** II only

**C** III only

**D** IV only

**E** I and IV only

**F** II and IV only

## Question 10

The logarithm function $\log(x)$ is defined for $x > 0$. For which values of $a$ does the equation

$$\log(x + a) = \log(x) + \log(a)$$

have a solution?

**A** $0 < a < 1$

**B** $a > 0$

**C** $a \geq 1$

**D** $a > 1$

**E** $a > 2$

## Question 11

Find a counter-example to the statement: if $p$ is prime then $2^p - 1$ is prime.

**A** $p = 2$

**B** $p = 3$

**C** $p = 7$

**D** $p = 11$

**E** $p = 13$

**F** $p = 23$

## Question 12

We say a function is **multiplicatively bounded** if and only if $f(xy) \leq f(x)f(y)$ and $f(0) > 0$.

Is $f(x) \geq 1$ a **necessary** condition for a multiplicatively bounded function?

**A** No - it is neither necessary nor sufficient

**B** No - it is sufficient but not necessary

**C** Yes - it is necessary

**D** Yes - it is necessary and sufficient

**E** It depends on the value of f(0)

## Question 13

For this question we assume our given function $\mathrm{f}(x)$ is differentiable and thus is (Riemann) integrable.

We say $\mathrm{f}(x)$ is odd if and only if $\mathrm{f}(-x) = -\mathrm{f}(x)$.

Which of the following conditions are **necessary** for $\mathrm{f}(x)$ to be an odd function?

I. $\mathrm{f}(x)$ is a constant function.

II. $\mathrm{f}(a - x) + \mathrm{f}(a + x) = \mathrm{f}(a)$, where $a$ is any real constant.

III. $\displaystyle\int_{-a}^{a} \mathrm{f}(x)\,\mathrm{d}x = 0$, where $a$ is some real constant.

IV. $\mathrm{f}'(x) > 0$.

**A** none of them

**B** I only

**C** II only

**D** III only

**E** IV only

**F** I and III only

**G** II and III only

**H** III and IV only

## Question 14

How many solutions are there to the equation

$$\tan(2x - x\sin x) = 1$$

for $0 < x < \pi$?

**A** 0

**B** 1

**C** 2

**D** 3

**E** 4

**F** 5

## Question 15

A discrete random variable $X$ takes non-negative values with distribution $\mathbb{P}(X = k) = ab^{-k}$, where $k \geq 0$ is an integer and $a > 0$ and $b > 1$ are real numbers.

Find $a$ in terms of $b$.

**A** $\frac{1}{b}$

**B** $\frac{1}{b-1}$

**C** $\frac{b}{b-1}$

**D** $b-1$

**E** $1-\frac{1}{b}$

## Question 16

Find the value of

$$\log\left(\frac{1}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{9}{4}\right) + \log\left(\frac{16}{5}\right) + \cdots + \log\left(\frac{n^2}{n+1}\right)$$

in the form $\log \mathrm{f}(n)$ where $\mathrm{f}(n)$ is some function of $n$.

**A** $\log\left(\dfrac{n!}{n+1}\right)$

**B** $\log\left(\dfrac{n^2}{n+1}\right)$

**C** $\log\left(\dfrac{n!}{(n+1)!}\right)$

**D** $\log\left(\dfrac{(n!)^2}{n+1}\right)$

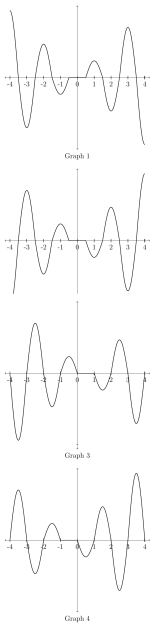
**E** $\log\left(\dfrac{(n+1)!}{n!}\right)$

## Question 17

We define the floor function $\lfloor x \rfloor$ to be equal to the largest integer less than or equal to $x$, so that $\lfloor 2 \rfloor = 2$ and $\lfloor \frac{10}{3} \rfloor = 3$.

Which of the following is a graph of

$$\mathrm{f}(x) = \lfloor x + \tfrac{1}{2} \rfloor \sin\left(\pi x + \tfrac{\pi}{2}\right)?$$




**A** Graph 1

**B** Graph 2

**C** Graph 3

**D** Graph 4

## Question 18

A sequence $u_k$ is defined such that $u_1 = a$, $u_2 = b$ and $u_{n+1} = u_n - u_{n-1}$ for $n \geq 2$.

Find the value of

$$\sum_{k=1}^{62} u_k$$

in terms of $a$ and $b$.

**A** $0$

**B** $a$

**C** $b$

**D** $b - a$

**E** $-a$

**F** $a + b$

## Question 19

A sequence of functions $\mathrm{f}_k(x)$ is defined such that $\mathrm{f}_1(x) = \alpha + \beta x$ and

$$\mathrm{f}_{n+1}(x) = \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_0^x x\,\mathrm{f}_n(x)\,\mathrm{d}x\right)$$

for $n \geq 1$. What is the value of $\displaystyle\sum_{k=1}^{\infty} \mathrm{f}_k(1)$?

**A** $\dfrac{4\alpha}{3} + 2\beta$

**B** $\dfrac{\alpha}{3} + 2\beta$

**C** $\dfrac{3\alpha}{4} + 2\beta$

**D** $\dfrac{4\alpha}{3} + \beta$

**E** $4\alpha + 2\beta$