



Daniel J Smith Custom TMUA Set A

Paper 1

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: Daniel J Smith

Question 1

Let

$$f(x) = a^{\cos x} + a^{-\cos x}$$

with $a > 0$ and real x

The difference between the maximum and minimum values of $f(x)$ is 4

Find the sum of the possible values of a

A 2

B 3

C 4

D 5

E 6

F $3\sqrt{2}$

Question 2

For non-zero real numbers a and b , consider the equation

$$\log_3(x - a) = \log_3 x + b$$

The number of real solutions x depends on a and b .

Which of the following is correct?

- A** Exactly one real solution if $a(1 - 3^b) > 0$ **and** no real solutions if $a(1 - 3^b) < 0$
- B** Exactly one real solution if $a(1 - 3^b) < 0$ **and** no real solutions if $a(1 - 3^b) > 0$
- C** Exactly two real solutions if $a(1 - 3^b) > 0$ **and** none otherwise
- D** No real solutions if $ab > 0$ **and** exactly one real solution if $ab < 0$
- E** The equation always has exactly one real solution for all non-zero a and b

Question 3

Fix $a > 0$ with $a \neq 1$

For $x > 0$ consider the equation

$$\log_a x = x$$

The number of positive solutions of this equation is

- A** exactly one if $0 < a < 1$, two if $1 < a < e^{1/e}$, one if $a = e^{1/e}$, and none if $a > e^{1/e}$
- B** exactly two if $0 < a < 1$, one if $a = 1$, and none if $a > 1$
- C** exactly one for all $a > 0$, $a \neq 1$
- D** none if $0 < a < 1$, one if $a = e^{1/e}$, and two if $a > e^{1/e}$
- E** none if $a > 1$, and exactly one if $0 < a < 1$

Question 4

Define functions g_1, g_2, g_3, g_4 on the real numbers by

$$\begin{aligned}g_1(x) &= \sin x \\g_2(x) &= \sin(\sin x) \\g_3(x) &= \sin(\sin(\sin x)) \\g_4(x) &= \sin(\sin(\sin(\sin x)))\end{aligned}$$

Let M_k be the maximum value of g_k for $k = 1, 2, 3, 4$

Which one of the following statements is true?

- A** $M_1 = M_2 = M_3 = M_4 = 1$
- B** $1 > M_1 > M_2 > M_3 > M_4 > 0$
- C** $1 > M_1 = M_2 = M_3 = M_4 > 0$
- D** $M_1 = 1$ and $1 > M_2 > M_3 > M_4 > 0$
- E** $M_2 = M_4$ and $M_1 = M_3$
- F** None of the above

Question 5

Consider

$$P(x) = (1 + x)^8(1 - 2x)^5$$

The coefficient of x^3 in $P(x)$ is equal to k times the coefficient of x^2 in $P(x)$

What is the value of k ?

A $-\frac{9}{4}$

B $-\frac{1}{4}$

C 0

D $\frac{1}{4}$

E $\frac{9}{4}$

F $-\frac{4}{3}$

Question 6

Let f be a polynomial defined for all real x , and define

$$J = \int_{-2}^3 (f(x) - f(|x|)) \, dx$$

Which of the following expressions is equal to J for all such polynomials f ?

A $-\int_0^2 f(t) \, dt$

B $\int_0^2 f(t) \, dt$

C $\int_{-2}^0 f(t) \, dt - \int_0^2 f(t) \, dt$

D 0

E $f(0)$

F $f(2) - f(0)$

G $\int_{-2}^0 f(t) \, dt$

Question 7

Let $a > 0$ and k be real numbers

Consider the cubic

$$g(x) = x^3 - 3ax + k$$

For which complete range of values of k does the equation $g(x) = 0$ have three distinct real roots?

A $-2a^{3/2} < k < 2a^{3/2}$

B $-2a^{3/2} \leq k \leq 2a^{3/2}$

C $-a^{3/2} < k < a^{3/2}$

D $-2a^3 < k < 2a^3$

E $k < -2a^{3/2}$ **or** $k > 2a^{3/2}$

F $-2a^{3/2} < k \leq 2a^{3/2}$

G $-2a^{3/2} \leq k < 2a^{3/2}$

Question 8

Consider the curve defined by

$$(|x| - 4)^2 + (y - 1)^2 = 1$$

How many distinct straight lines passing through the origin are tangents to this curve?

- A** 0
- B** 1
- C** 2
- D** 3
- E** 4
- F** 5

Question 9

For a real constant k , define

$$A(k) = \int_{-1}^2 |x - k| \, dx$$

For which value of k is $A(k)$ minimised?

A -1

B $\frac{1}{2}$

C 1

D $\frac{3}{2}$

E 2

F 0

Question 10

Let a and b be real numbers with $a + b = 1$

What is the least possible value of

$$(a^2 + b)(b^2 + a) \quad ?$$

A $\frac{9}{16}$

B $\frac{1}{4}$

C $\frac{1}{2}$

D $\frac{3}{4}$

E 1

Question 11

Let k be an integer with $0 \leq k \leq 10$

Consider the quadratic equation

$$x^2 - 2kx + (k^2 - 3k + 2) = 0$$

For how many values of k does the equation have two distinct positive real roots?

A 6

B 7

C 8

D 9

E 10

Question 12

The interior angle of a regular polygon with n sides exceeds the interior angle of a regular polygon with m sides by exactly 18°

How many ordered pairs of integers (n, m) with $n \geq 3$ and $m \geq 3$ satisfy this condition?

- A** 0
- B** 1
- C** 2
- D** 3
- E** 7
- F** infinitely many

Question 13

Let c be a real constant.

Consider the equation

$$|\sin x| + |\cos x| = c$$

for $0 \leq x < 2\pi$

Which of the following gives the number of distinct solutions as a function of c ?

- A** If $0 < c < 1$, there are 8 solutions; if $1 < c < \sqrt{2}$ there are 4 solutions; if $c = 1$ **or** $c = \sqrt{2}$ there are 2 solutions; otherwise there are 0 solutions
- B** If $0 < c < 1$, there are 0 solutions; if $c = 1$ there are 4 solutions; if $1 < c < \sqrt{2}$ there are 8 solutions; if $c = \sqrt{2}$ there are 4 solutions; otherwise there are 0 solutions
- C** If $0 < c < 1$, there are 0 solutions; if $c = 1$ there are 2 solutions; if $1 < c < \sqrt{2}$ there are 4 solutions; if $c = \sqrt{2}$ there are 8 solutions; otherwise there are 0 solutions
- D** If $0 < c < 1$, there are 4 solutions; if $c = 1$ there are 8 solutions; if $1 < c < \sqrt{2}$ there are 4 solutions; if $c = \sqrt{2}$ there are 2 solutions; otherwise there are 0 solutions
- E** If $0 < c < 1$, there are 0 solutions; if $1 \leq c \leq \sqrt{2}$ there are 8 solutions; otherwise there are 0 solutions
- F** If $0 < c < \sqrt{2}$ there are 8 solutions; if $c = \sqrt{2}$ there are 4 solutions; if $c > \sqrt{2}$ there are 0 solutions

Question 14

Let a, b, c be real constants with $a > 0$ and $c \neq 0$.

Suppose the inequality $ax^2 + bx + c > 0$ has solution set

$$x < r \quad \text{or} \quad x > s$$

where $r < s$

In terms of r, s and c , what is the solution set of

$$a(cx)^2 + b(cx) + c > 0 \quad ?$$

A $x < cr \text{ or } x > cs$

B $x < \frac{r}{c} \text{ or } x > \frac{s}{c}$

C $x < cs \text{ or } x > cr$

D $x < \frac{s}{c} \text{ or } x > \frac{r}{c}$

E If $c > 0$: $x < \frac{r}{c} \text{ or } x > \frac{s}{c}$; if $c < 0$: $x < \frac{s}{c} \text{ or } x > \frac{r}{c}$

F If $c > 0$: $\frac{r}{c} < x < \frac{s}{c}$; if $c < 0$: $\frac{s}{c} < x < \frac{r}{c}$

G If $c > 0$: $x < \frac{r}{c} \text{ or } x > \frac{s}{c}$; if $c < 0$: $x < \frac{r}{c} \text{ or } x > \frac{s}{c}$

Question 15

A sequence (u_n) is defined by $u_1 > 0$ and

$$u_{n+1} = \frac{u_n}{1 + u_n}$$

for all integers $n \geq 1$

Which of the following is equal to $\sum_{n=1}^{\infty} u_n$?

A u_1

B $\frac{u_1}{1-u_1}$

C $\frac{1}{u_1}$

D $1 + u_1$

E $\frac{1}{1+u_1}$

F The series does not converge for any $u_1 > 0$

Question 16

Let

$$f(x) = x^3 - 3x + 1$$

How many distinct real solutions does the equation $f(x) = 0$ have?

- A** 0
- B** 1
- C** 2
- D** 3
- E** 4
- F** Infinitely many

Question 17

Let a and b be positive integers and define a sequence v_n by

$$v_1 = a, \quad v_2 = b, \quad v_{n+2} = 2v_{n+1} - v_n \quad \text{for } n \geq 1$$

Suppose the highest common factor of a and b is d

Which of the following statements must be true?

- A** The highest common factor of v_{10} and v_{11} is 1
- B** v_n is divisible by d only for even n
- C** The highest common factor of v_n and v_{n+1} is d for all $n \geq 1$
- D** For all $n \geq 1$, v_n is divisible by $2d$
- E** If $a \neq b$ then v_n is never divisible by d

Question 18

How many solutions are there to the equation

$$2 \sin x = |\cos x|$$

for $0^\circ \leq x < 360^\circ$, and what is the sum of all the solutions?

- A** Number of solutions = 2, Sum of solutions = 180°
- B** Number of solutions = 2, Sum of solutions = 360°
- C** Number of solutions = 2, Sum of solutions = 720°
- D** Number of solutions = 4, Sum of solutions = 360°
- E** Number of solutions = 4, Sum of solutions = 540°
- F** Number of solutions = 4, Sum of solutions = 720°
- G** Number of solutions = 6, Sum of solutions = 900°
- H** Number of solutions = 8, Sum of solutions = 1440°

Question 19

A circle has centre O and radius 5

Points A and B lie on the circumference

The area of triangle AOB is 10

What is the greatest possible length of chord AB ?

A $4\sqrt{5}$

B $2\sqrt{5}$

C $5\sqrt{2}$

D 8

E 6

F $7\sqrt{3}$

Question 20

Let $f(x) = x^3 - 3x$

Consider the following three curves:

- (1) $y = f(x)$
- (2) $y = f(x) + 2$
- (3) the curve $y = f(x)$ reflected in the x -axis

The trapezium rule is used to estimate the signed area $\int_{-1}^1 y \, dx$ for each curve using the points $x = -1$, $x = 0$ and $x = 1$

Select the row that correctly describes whether the trapezium rule gives an overestimate, an underestimate, or an exact value for each curve

- A** Curve (1) exact; Curve (2) exact; Curve (3) exact
- B** Curve (1) exact; Curve (2) exact; Curve (3) overestimate
- C** Curve (1) exact; Curve (2) underestimate; Curve (3) exact
- D** Curve (1) underestimate; Curve (2) exact; Curve (3) exact
- E** Curve (1) overestimate; Curve (2) exact; Curve (3) underestimate
- F** Curve (1) underestimate; Curve (2) underestimate; Curve (3) overestimate