



Daniel J Smith Custom TMUA Set A

Paper 2

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: Daniel J Smith

Question 1

For each positive integer n , define

$$E_n = n^4 + 4$$

For which one of the following statements is $n = 3$ a counterexample?

- A** For all positive integers n , E_n is divisible by $n^2 - 2n + 2$
- B** For all positive integers n , E_n is divisible by $n^2 + 2n + 2$
- C** For all positive integers n , E_n is **not** divisible by 17
- D** For all positive integers n , E_n is **not** divisible by 3
- E** For all positive integers n , E_n is divisible by 5
- F** For all positive integers n , E_n has remainder 1 when divided by 4

Question 2

A student is asked to prove the statement:

if u **and** v are real numbers such that $u^2 + v^2 = 0$, **then** $u = 0$ **and** $v = 0$

They write the following argument:

I. For all real w , $w^2 \geq 0$

II. So $u^2 \geq 0$ **and** $v^2 \geq 0$

III. If the sum of two real numbers is 0 and each is **not** negative, then each number is 0

IV. Therefore $u^2 = 0$ **and** $v^2 = 0$

V. Hence $u = 0$ **and** $v = 0$

Which of the following best describes this argument?

- A** It is a correct direct proof of the statement.
- B** It is incorrect, because line I is false.
- C** It is incorrect, because line III is false.
- D** It is incorrect, because line IV does not follow from the previous lines.
- E** It is incorrect, because the reasoning would only work if u **and** v were positive.
- F** It is a correct proof of the converse, but **not** of the original statement.

Question 3

We say a sequence of real numbers x_n is a *Cauchy sequence* if

for all $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $|x_m - x_n| < \varepsilon$

Which one of the following statements is true if and only if x_n is **not** a Cauchy sequence?

- A** There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exist $m, n \geq N$ with $|x_m - x_n| \geq \varepsilon$
- B** For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and there exists $m, n \geq N$ with $|x_m - x_n| \geq \varepsilon$
- C** There exists $N \in \mathbb{N}$ such that for all $\varepsilon > 0$ and all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$
- D** For every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there exists $m, n \geq N$ with $|x_m - x_n| < \varepsilon$
- E** There exists $\varepsilon > 0$ and there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$
- F** For every $N \in \mathbb{N}$ there exists $\varepsilon > 0$ such that for all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$
- G** There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ and for all $m, n \leq N$ we have $|x_m - x_n| \geq \varepsilon$
- H** There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exist $m, n \geq N$ with $|x_m - x_n| < \varepsilon$

Question 4

Let S be the statement about integers n :

(*) For every integer n there exists an integer m such that $m > n$ **and** m is even

Which of the following statements is **logically equivalent** to the **negation** of (*)?

- A** For some integer n there exists an integer m such that $m > n$ **and** m is odd.
- B** For some integer n , **if** an integer m is greater than n **then** m is even.
- C** For every integer n , **if** n is even **then** there is no integer $m > n$ that is even.
- D** It is **not** the case that there exists an integer m which is greater than every integer n **and** is even.
- E** For every integer m there exists an integer $n < m$ such that m is odd.
- F** There exists an integer n such that there is exactly one even integer greater than n .
- G** There exists an integer n such that for all integers m , **if** $m > n$ **then** m is **not** even.

Question 5

Consider the statement about integers n :

(*) For every integer n , **if** n is even **then** $n^2 + 2n$ is divisible by 4

Which of the following statements has the same **truth value** as (*)?

I: For all integers n , **if** $n^2 + 2n$ is **not** divisible by 4 **then** n is odd

II: There exists an integer n which is even **and** for which $n^2 + 2n$ is divisible by 4

III: For some integer n , **if** n is even **then** $n^2 + 2n$ is divisible by 4

A I only

B II only

C III only

D I and II only

E I and III only

F II and III only

G I, II and III

H none of them

Question 6

A statement S is given:

For all real numbers x , **if** $x^2 \geq 1$ **then** $x^4 \geq x^2$

A student wishes to prove S by contradiction and writes the following argument:

Assume that there exists a real number x such that $x^2 \geq 1$ **and** $x^4 < x^2$

I. Then $x^4 - x^2 < 0$

II. So $x^2(x^2 - 1) < 0$

III. So $x^2 - 1 < 0$

IV. So $x^2 < 1$, which contradicts $x^2 \geq 1$

Therefore the original statement S is concluded to be true.

Which of the following best describes this argument?

- A** It is completely correct.
- B** It is incorrect, but it would be correct if written in the reverse order.
- C** It is incorrect, and the student has actually shown that S is false.
- D** It is incorrect because line II does not follow from line I.
- E** It is incorrect because line III does not follow from line II.
- F** It is incorrect because the student has **not** really used proof by contradiction.

Question 7

For real numbers a and b , consider the statement $S(a, b)$:

if $ab < 0$ **then** $a^2 + b^2 > 0$

Which of the following descriptions is correct?

- A** $S(a, b)$ is false for some real a, b because $a^2 + b^2$ can be negative
- B** $S(a, b)$ is false for some real a, b because $a^2 + b^2 = 0$ can occur when $ab < 0$
- C** $S(a, b)$ is false for some real a, b because $ab < 0$ forces $a^2 + b^2 = 0$
- D** $S(a, b)$ is true for all real a, b , and the condition $ab < 0$ is **necessary** for $a^2 + b^2 > 0$
- E** $S(a, b)$ is true for all real a, b , and the condition $ab < 0$ is **sufficient and necessary** for $a^2 + b^2 > 0$
- F** $S(a, b)$ is true for all real a, b , and the condition $ab < 0$ is **sufficient** but **not necessary** for $a^2 + b^2 > 0$
- G** $S(a, b)$ is true for all real a, b , and the condition $ab < 0$ is neither **necessary** nor **sufficient** for $a^2 + b^2 > 0$

Question 8

Consider the statement:

For every positive integer n there exists a positive integer k such that for all positive integers m , the number $n(m + k)$ is **not** a square

Which one of the following is the **negation** of this statement?

- A** For every positive integer n there exists a positive integer k such that for every positive integer m , the number $n(m + k)$ is a square.
- B** There exists a positive integer n such that for every positive integer k **and** every positive integer m , the number $n(m + k)$ is **not** a square.
- C** There exists a positive integer n such that there exists a positive integer k for which $n(m + k)$ is a square for all positive integers m .
- D** For every positive integer n , for every positive integer k , there exists a positive integer m such that $n(m + k)$ is **not** a square.
- E** There exists a positive integer n such that for every positive integer k there exists a positive integer m such that $n(m + k)$ is a square.

Question 9

Let $f(x) = ax^2 + bx + c$ be a quadratic function with real coefficients such that $f(x) \geq 0$ for all real x . Suppose that for every real x ,

$$f(x+1) - f(x) \geq 2x + 3$$

What is the smallest possible value of $f(0)$?

A 0

B $\frac{1}{4}$

C 1

D 2

E $\frac{9}{4}$

F 4

G $\frac{25}{4}$

Question 10

Let C be the circle

$$x^2 + y^2 = 9$$

Let P be the point $(a, 0)$ where $a > 3$

The two tangents from P to C touch the circle at T_1 and T_2

What is the area of triangle PT_1T_2 in terms of a ?

A $3\sqrt{a^2 - 9}$

B $\frac{3(a^2-9)^{3/2}}{a^2}$

C $\frac{9}{a}\sqrt{a^2 - 9}$

D $\frac{a}{3}\sqrt{a^2 - 9}$

E $\frac{a^2-9}{3}$

F $\frac{a^2-9}{a}$

G $\frac{9}{\sqrt{a^2-9}}$

Question 11

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x

Define

$$f(x) = \lfloor 2x \rfloor - \lfloor x \rfloor$$

for all real x

The value of $\int_0^2 f(x) \, dx$ is

A 0

B $\frac{1}{2}$

C 1

D $\frac{3}{2}$

E 2

F 3

G 4

Question 12

A sequence (a_n) is defined by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n + 4}{\sqrt{a_n + 1}}$$

Which of the following statements is true?

- A** (a_n) is strictly decreasing for all n
- B** (a_n) is bounded above but **not** bounded below
- C** (a_n) is bounded below by 1 and bounded above by 4
- D** (a_n) is bounded below by 2 and bounded above by 4
- E** (a_n) is bounded below by 1 but **not** bounded above
- F** $a_n > 4$ for all $n \geq 2$
- G** The sequence (a_n) converges to 4

Question 13

Let P and Q be statements

Suppose exactly one of (**if P then Q**) and (**if Q then P**) is true

Suppose further that (P **or** Q) is true

Which statement must be true?

- A** (P **and** Q) is true
- B** Exactly one of P and Q is true
- C** Both P and Q are false
- D** (**if $\neg P$ then $\neg Q$**) is true
- E** (P **if and only if** Q) is true

Question 14

Let x be a real number

Which one of the following statements is a **sufficient** condition for exactly two of the other four statements?

A $x \geq 4$

B $x > 1$

C $x^2 \geq 10$

D $x \geq (x - 4)^2$

E $0 \leq x \leq 3$

Question 15

A non-empty finite set S of whole numbers is called *linked* **if and only if**

for every a in S , there exists b in S with $b \neq a$ such that $\gcd(a, b) > 1$

where $\gcd(a, b)$ denotes the greatest common divisor (also known as the highest common factor) of integers a and b

Which of the following is true **if and only if** S is **not** linked?

- A** For every a in S , there exists b in S with $b \neq a$ such that $\gcd(a, b) = 1$
- B** For every a in S , there exists a prime factor of a that divides every element of S
- C** There exists a in S such that for every b in S with $b \neq a$ we have $\gcd(a, b) > 1$
- D** There exists a in S such that for every b in S with $b \neq a$ we have $\gcd(a, b) = 1$
- E** There exists a prime p such that p divides exactly one element of S

Question 16

Consider the statement:

if $0 \leq f(x) \leq 1$ for all real x with $0 \leq x \leq 1$, **then**

$$(*) \quad \int_0^1 f(x) \, dx \leq \int_0^1 (f(x))^2 \, dx$$

Which of the following functions is a *counterexample* to $(*)$?

A $f(x) = 2x$

B $f(x) = x^2 + \frac{1}{2}$

C $f(x) = \frac{1}{2}$

D $f(x) = 1 + x$

E $f(x) = 0$

F $f(x) = 5x(1 - x)$

G $f(x) = \sqrt{x} + \frac{1}{2}$

H $f(x) = 1$

Question 17

Consider the claim:

For all integers a, b, c , **if** $ab = ac$ **then** $b = c$

A student's argument is as follows:

I. Suppose $ab = ac$

II. Then $ab - ac = 0$

III. So $a(b - c) = 0$

IV. Therefore $b - c = 0$

V. Hence $b = c$

Which of the following best describes this argument?

- A** The argument is completely correct.
- B** The argument is incorrect, and the first error is on line I
- C** The argument is incorrect, and the first error is on line II
- D** The argument is incorrect, and the first error is on line III
- E** The argument is incorrect, and the first error is on line IV
- F** The argument is incorrect, and the first error is on line V

Question 18

Let f be a function defined for all real x , and assume the integral exists

Which one of the following is a **sufficient** condition for

$$\int_{-2}^4 f(x) \, dx = 0$$

- A** $f(1) = 0$
- B** $f(-2) = f(4) = 0$
- C** $f(-x) = -f(x)$ for all real x
- D** $f(x+1) = -f(1-x)$ for all real x
- E** $f(x+2) = -f(2-x)$ for all real x
- F** $f(x-1) = -f(1-x)$ for all real x

Question 19

Consider the two statements about a real-valued function f :

(A) For all real numbers x there exists a real number y such that $y > x$ **and** $f(y) < f(x)$

(B) There exists a real number y such that for all real numbers x with $x < y$, $f(y) < f(x)$

Which of the following is true?

- A** Both (A) **and** (B) are always true for any function f .
- B** For some functions f , (A) is true **and** (B) is true, but **not** for all f .
- C** There is a function f for which (A) is true **and** (B) is false, but **not** the other way round.
- D** There is a function f for which (B) is true **and** (A) is false, but **not** the other way round.
- E** For every function f , (A) **and** (B) are either both true or both false.
- F** There is no function f for which (A) is true.

Question 20

The positive real numbers $p \times 10^{-4}$, $q \times 10^{-2}$ and $r \times 10^{-1}$ are each in standard form, and

$$(p \times 10^{-4}) + (q \times 10^{-2}) = (r \times 10^{-1})$$

How many of the following statements must be true?

I: $q > 1$

II: $r > 1$

III: $q < r$

IV: $p < r$

A 1 of them

B 2 of them

C 3 of them

D 4 of them

E None of them