TMUA

COMMUNITY PAPERS

Daniel J Smith Custom TMUA alpha

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: Daniel J Smith

Question 1

$PQRS$ is a quadrilateral, labelled anticlockwise.

Which one of the following is a **sufficient** but **not necessary** condition for $PQRS$ to be a parallelogram?

A PR and QS bisect each other

B $PQ = QR = RS = SP$

C $\angle P + \angle R = 180^\circ$ **and** $\angle Q + \angle S = 180^\circ$

D PR bisects QS

Question 2

A student chooses two distinct real numbers x and y with $0 < x < y < 1$.

The student then attempts to draw a triangle ABC with

$$\begin{aligned} AB &= 1, \\ \cos A &= x, \\ \cos B &= y \end{aligned}$$

Which of the following statements is/are correct?

I: For some choice of x and y, there is no such triangle.

II: For some choice of x and y, there is exactly one such triangle.

III: For some choice of x and y, there are exactly two such triangles.

Note that congruent triangles are considered to be the same.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Question 3

We say an integer x *divides* an integer y and write $x \mid y$ if there exists an integer k such that $y = kx$

Let a, b, n be positive integers. Let $(*)$ the statement

$$(*) \qquad n \mid ab$$

Let C be the statement:

For every prime p with $p \mid n$, either $p \mid a$ or $p \mid b$

This condition C is:

A **necessary** but **not sufficient** for $(*)$

B **sufficient** but **not necessary** for $(*)$

C **necessary** and **sufficient** for $(*)$

D **neither necessary nor sufficient** for $(*)$

Question 4

Let $n > 1$ be an integer. Consider the property $P(n)$:

For all integers a, b, **if** $n \mid ab$ **then** $n \mid a$ **or** $n \mid b$.

Which of the following is correct?

A $P(n)$ holds **if and only if** n is prime

B $P(n)$ holds **if and only if** n is squarefree

C $P(n)$ is **sufficient** but **not necessary** for n to be prime

D $P(n)$ holds **if and only if** n is a prime power

E $P(n)$ holds **if and only if** n has at most two distinct prime factors

Question 5

Let $P(n)$ be a statement about integers with the following properties:

$P(1)$ and $P(2)$ are true;

for all $n \geq 1$ the following statement is true: $(P(n)$ **and** $P(n+1))$ is **sufficient** for $P(n+2)$

What is the strongest conclusion that necessarily follows?

A $P(n)$ is true for all $n \geq 1$

B $P(n)$ is true for all sufficiently large n, but not necessarily for all n

C $P(n)$ is true for infinitely many n, but not necessarily for all n

D $P(n)$ is true for all odd n but not necessarily for all even n

E $P(n)$ is true for all even n but not necessarily for all odd n

F No general conclusion can be drawn beyond $P(1)$ and $P(2)$

Question 6

Let real numbers x and y satisfy $|x| + |y| = 1$. What is the maximum possible value of $x - 2y$?

A 2

B $\sqrt{5}$

C $\frac{5}{2}$

D $\frac{3}{2}$

E $\sqrt{10}$

Question 7

Let P and Q be statements. Suppose exactly one of **if** P **then** Q and **if** Q **then** P is true, and also P **or** Q is true. Which statement must be true?

A P **and** Q is true

B Exactly one of P and Q is true

C Both P and Q are false

D **if** $\neg P$ **then** $\neg Q$ is true

E P **if and only if** Q is true

Question 8

For $a > 0$ with $a \neq 1$, consider the equation $\log_a x = x$ for $x > 0$. The number of positive solutions is

A exactly one if $0 < a < 1$, two if $1 < a < e^{1/e}$, one if $a = e^{1/e}$, and none if $a > e^{1/e}$

B exactly two if $0 < a < 1$, one if $a = 1$, and none if $a > 1$

C exactly one for all $a > 0$, $a \neq 1$

D none if $0 < a < 1$, one if $a = e^{1/e}$, and two if $a > e^{1/e}$

E none if $a > 1$, and exactly one if $0 < a < 1$

Question 9

For real parameter t, consider the equation

$$\log_3 x + \log_x 3 = t$$

with $x > 0$ and $x \neq 1$. The number of positive solutions x is

A two if $|t| > 2$, one if $|t| = 2$, and zero if $|t| < 2$

B one if $t > 0$, and zero if $t < 0$

C zero if $t > 2$, one if $t = 2$, and two if $0 < t < 2$

D one for all real t

E two if $t < -2$, zero if $t > -2$, and one if $t = -2$

Question 10

$ABCD$ is a quadrilateral, labelled anticlockwise.

Which one of the following is a **sufficient** but **not necessary** condition for $ABCD$ to be a rectangle?

A Both pairs of opposite sides are parallel **and** one interior angle is 90°

B All four sides are equal **and** one interior angle is 90°

C The diagonals are equal

D Opposite angles sum to 180°

Question 11

Given that $\frac{x^2}{4} + y^2 = 1$, what is the greatest possible value of $3x + 4y$?

A $\sqrt{13}$

B 5

C $\sqrt{50}$

D $2\sqrt{13}$

E 7

Question 12

Fix distinct real numbers $a \neq b$ with $a < b$. Consider the equation

$$|x - a| + |x - b| = |a - b|$$

Which of the following describes the complete solution set for x?

A $x = \frac{a+b}{2}$

B $x = a$ **or** $x = b$

C all real x with $a \leq x \leq b$

D all real x

E no real solutions

Question 13

Consider the statement $(*)$ about a real number x:

There exists a real number y such that $|x + y| < y$.

Which one of the following describes all real values of x for which $(*)$ is true?

A no values of x

B exactly one value of x

C all real x with $x < 0$

D all real x with $x > 0$

E all real values of x

F all real x except exactly one value

Question 14

Let a be a real number. The equation

$$|x-1|+|x+1|=a$$

has exactly k distinct real solutions for x.

Which one of the following is the complete list of all possible values of k as a varies over the real numbers?

A 0 only

B $0, 1$

C $0, 2$

D $1, 2$

E $0, 1, 2$

F 2 only

Question 15

Consider the following statement about a polynomial $p(x)$ with real coefficients:

$(*)$ If $p(n)$ is even for every integer n, then every coefficient of $p(x)$ is even.

Which one of the following is a *counterexample* to $(*)$?

A $p(x) = x^2 + x$

B $p(x) = 2x^2 + 4$

C $p(x) = 2x^2 + x$

D $p(x) = 4x^3 + 2x$

Question 16

Consider the following statement about irrational numbers:

($*$) If x and y are irrational numbers, then $x + y$ is irrational.

Which of the following pair(s) (x, y) provide(s) a *counterexample* to ($*$)?

I: $x = \sqrt{2}$, $y = \sqrt{2}$

II: $x = \sqrt{2}$, $y = 3$

III: $x = 1 + \sqrt{2}$, $y = 1 - \sqrt{2}$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Question 17

We say a real function $f(x)$ is *continuous* at a point $x_0 \in \mathbb{R}$ if

for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$

Which one of the following statements is true if and only if f is **not** continuous at x_0?

A There exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists x with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$

B For every $\varepsilon > 0$ and every $\delta > 0$ there exists x with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$

C There exists $\varepsilon > 0$ and $\delta > 0$ such that for all x with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| \geq \varepsilon$

D There exists $\varepsilon > 0$ such that for every $\delta > 0$ we have $|x - x_0| \geq \delta$ or $|f(x) - f(x_0)| \geq \varepsilon$

E For every $\varepsilon > 0$ and every $\delta > 0$ there exists x with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| < \varepsilon$

F There exists $\delta > 0$ such that for all $\varepsilon > 0$ and all x with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$

G For every $\varepsilon > 0$ there exists $\delta > 0$ such that there exists x with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$

H There exists $\varepsilon > 0$ such that for every $\delta > 0$ if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| \geq \varepsilon$

Question 18

Let $f(x)$, $g(x)$ and $h(x)$ be real functions and consider the statement

$(*)$ If $x > 0$ and $f(x) \leq 0$ then $g(x) \neq 1$ or $h(x) \geq -1$.

Which of the following statements is the *contrapositive* of $(*)$?

A If $g(x) = 1$ and $h(x) < -1$ then $x \leq 0$ or $f(x) > 0$.

B If $g(x) = 1$ or $h(x) < -1$ then $x \leq 0$ or $f(x) > 0$.

C If $x \leq 0$ or $f(x) > 0$ then $g(x) = 1$ and $h(x) < -1$.

D If $g(x) \neq 1$ or $h(x) \geq -1$ then $x > 0$ and $f(x) \leq 0$.

E If $x > 0$ and $f(x) \leq 0$ then $g(x) = 1$ and $h(x) < -1$.

Question 19

We say a sequence of real numbers x_n is *a Cauchy sequence* if

for all $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|x_n - x_m| < \varepsilon$

Which one of the following statements is true if and only if x_n is **not** a Cauchy sequence?

A There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exist $m, n \geq N$ with $|x_m - x_n| \geq \varepsilon$

B For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and there exists $m, n \geq N$ with $|x_m - x_n| \geq \varepsilon$

C There exists $N \in \mathbb{N}$ such that for all $\varepsilon > 0$ and all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$

D For every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there exists $m, n \geq N$ with $|x_m - x_n| < \varepsilon$

E There exists $\varepsilon > 0$ and there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$

F For every $N \in \mathbb{N}$ there exists $\varepsilon > 0$ such that for all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$

Question 20

Let $f(x)$, $g(x)$, $h(x)$ and $k(x)$ be real functions and consider the statement

($*$) If $0 < x < 1$ and $f(x) > g(x)$, then $h(x) = 0$ or $k(x) \geq 5$.

Which of the following statements is the *contrapositive* of ($*$)?

A If $h(x) \neq 0$ or $k(x) < 5$ then $x \leq 0$ or $x \geq 1$ or $f(x) \leq g(x)$.

B If $h(x) \neq 0$ and $k(x) < 5$ then $x \leq 0$ or $x \geq 1$ and $f(x) \leq g(x)$.

C If $h(x) \neq 0$ and $k(x) < 5$ then $x \leq 0$ or $x \geq 1$ or $f(x) \leq g(x)$.

D If $x \leq 0$ or $x \geq 1$ or $f(x) \leq g(x)$ then $h(x) \neq 0$ and $k(x) < 5$.

E If $h(x) = 0$ or $k(x) \geq 5$ then $0 < x < 1$ and $f(x) > g(x)$.