TMUA
COMMUNITY PAPERS

Gerard Sayson TMUA Mock Paper 1

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: Gerard Sayson

Question 1

Find the area between the curve $y = x \ln x$ and the x-axis from $x = 0$ to $x = 2$.

A $2 \ln 2 - 1$

B $\ln 2 + 1$

C $2 \ln 2 + 1$

D $2 \ln 2 - \frac{1}{2}$

Question 2

It is given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 + a^3}{x^2 - ax + a^2}$$

where a is a constant, and $y = 2$ when $x = 1$.

Find an expression for y.

A $y = \frac{x^2}{2} + ax + \frac{3}{2} - a$

B $y = \frac{x^2}{2} - ax + \frac{3}{2} + a$

C $y = \frac{x^2}{2} + \frac{3}{2}$

D $y = ax + 2 - a$

E $y = \frac{x^2}{2} + ax + 2$

Question 3

The real roots of the equation

$$4^x - 3 \times 2^{x+1} + 5 = 0$$

are 2^α and 2^β. What quadratic expression has roots $\dfrac{\alpha}{\alpha+\beta}$ and $\dfrac{\beta}{\alpha+\beta}$?

A $x^2 - x$

B $x^2 - 6x + 5$

C $36x^2 - 36x + 5$

D $x^2 - x \log_2 5$

E $x^2 - x + \log_2 5$

Question 4

The function f is given as $\mathrm{f}(x) = x^2 + 6x + 12$.

Let g be such that its graph is the clockwise rotation of the graph of $y = \mathrm{f}(x)$ by 135° about the origin.

What equation is satisfied by $\mathrm{g}(y) = x$?

A $y^2 + 6y - x + 12 = 0$

B $x^2 + 2xy - 7\sqrt{2}\,x + y^2 - 5\sqrt{2}\,y + 24 = 0$

C $x^2 - 2xy - 5\sqrt{2}\,x + y^2 + 7\sqrt{2}\,y + 24 = 0$

D $x^2 - 2xy + 7\sqrt{2}\,x + y^2 - 5\sqrt{2}\,y + 24 = 0$

Question 5

Use the trapezium rule involving four points equally spaced apart to approximate the integral

$$\int_0^{2\pi} \sqrt{1+\sin^2 x}\,\mathrm{d}x$$

and determine whether there is an overestimate or underestimate.

A $\dfrac{2\pi}{3}\left(1+\sqrt{1+\dfrac{\sqrt{3}}{2}}+\sqrt{1-\dfrac{\sqrt{3}}{2}}\right)$, and there is an underestimate

B $\dfrac{2\pi}{3}\left(1+\sqrt{1+\dfrac{\sqrt{3}}{2}}+\sqrt{1-\dfrac{\sqrt{3}}{2}}\right)$, and there is an overestimate

C $\dfrac{2\pi\left(1+\sqrt{7}\right)}{3}$, and there is an underestimate

D $\dfrac{2\pi\left(1+\sqrt{7}\right)}{3}$, and there is an overestimate

E $\pi\left(1+\sqrt{2}\right)$, and there is an underestimate

F $\pi\left(1+\sqrt{2}\right)$, and there is an overestimate

Question 6

Points A and B lie on a circle with centre O and radius r. The line segment AB is a diameter of the circle. Point C lies on the circumference of the circle such that the angle CAB is 30°. A tangent to the circle at C intersects AB extended at a point D, with B between A and D.

Let S_1 be the area of triangle ABC and S_2 be the area of triangle OCD. Which of the following statements is true?

A $S_1 = 2S_2$

B $2S_1 = S_2$

C $S_1 = S_2$

D $S_1 = \sqrt{3}\,S_2$

E $\sqrt{3}\,S_1 = S_2$

Question 7

A coin is known to be either fair or biased, in which case the probability of heads is 0.6. Before any tosses are made, the probability that the coin is biased is p, where $0 \leq p \leq 1$.

The coin is tossed 5 times and lands on heads exactly 3 times.

What is the probability that the coin is biased?

A $\dfrac{3456p}{3125 + 331p}$

B $\dfrac{3125(1-p)}{3125 + 331p}$

C $\dfrac{216}{625}$

D $\dfrac{216}{625}p$

E $\dfrac{3456p}{8125 - 4669p}$

Question 8

240 people are divided into three groups A, B and C. The ratio of people in group A to that in group C is $1 : 3$, and group B would occupy a sector with angle $60°$ in a pie chart containing groups A, B and C.

How many people are there in group A?

A 40

B 50

C 60

D 70

E 150

Question 9

The region R is bounded by the curves $y = \dfrac{\sqrt{2\ln x}}{x}$ and $y = \dfrac{\ln x}{x}$.

The formula

$$V = \int_a^b \pi \left[\mathrm{f}(x)\right]^2 \, \mathrm{d}x$$

can be used to find the volume generated by rotating the area under the curve from $x = a$ to $x = b$ about the x-axis. Determine the exact volume generated by rotating R about the x-axis through 2π radians.

A $2\pi - 10\pi\mathrm{e}^{-2}$

B $4\pi\mathrm{e}^{-2} - \pi$

C $4\pi\mathrm{e}^{-2}$

D $4\pi\mathrm{e}^{-4}$

Question 10

A right pyramid has a square base with a side length of $\sqrt{6}$ cm. Given that the volume of the pyramid is

$$V = \frac{16}{3 - \sqrt{5}} \text{ cm}^3$$

find the exact vertical height of the pyramid in cm.

[The volume of a pyramid is given by $\frac{1}{3}Ah$, where A is the area of its base and h is its height.]

A $24 + 8\sqrt{5}$

B $6 + 8\sqrt{5}$

C $6 - 2\sqrt{5}$

D $6 + 2\sqrt{5}$

Question 11

A continuous dataset is partitioned into three classes:

$$0 \leq x < 10, \quad 10 \leq x < k, \quad k \leq x < 40,$$

where $10 < k < 40$.

The frequency densities of the three classes are in the ratio $1 : 2 : 1$ respectively. An estimate for the median of the data calculated using linear interpolation is exactly 20.

Find the value of k.

A 15

B 20

C 25

D 30

E 35

Question 12

Find the total length of the intervals in the range $0 \le x \le 4\pi$ for which

$$(\sin x - \cos x)(2\cos x - 1) \ge 0.$$

A $\frac{\pi}{4}$

B $\frac{\pi}{3}$

C $\frac{\pi}{2}$

D $\frac{2\pi}{3}$

E π

Question 13

Two geometric sequences (u_n) and (v_n) have the same first term a, where $a \neq 0$. One sequence has a common ratio of r and the other has a common ratio of r^2, where $|r| < 1$.

The sum to infinity of (u_n) is twice the sum to infinity of (v_n).

A new sequence is defined by $w_n = u_n^2$ for all $n \geq 1$. What is the sum to infinity of (w_n) in terms of a?

A $\dfrac{4a^2}{3}$

B $\dfrac{16a^2}{15}$

C $\dfrac{16a^2}{9}$

D $\dfrac{4a^2}{5}$

E $\dfrac{8a^2}{3}$

Question 14

Determine the exact value of $\tan 1° \tan 2° \ldots \tan 89°$.

A $\frac{1}{2}$

B $\frac{\sqrt{3}}{3}$

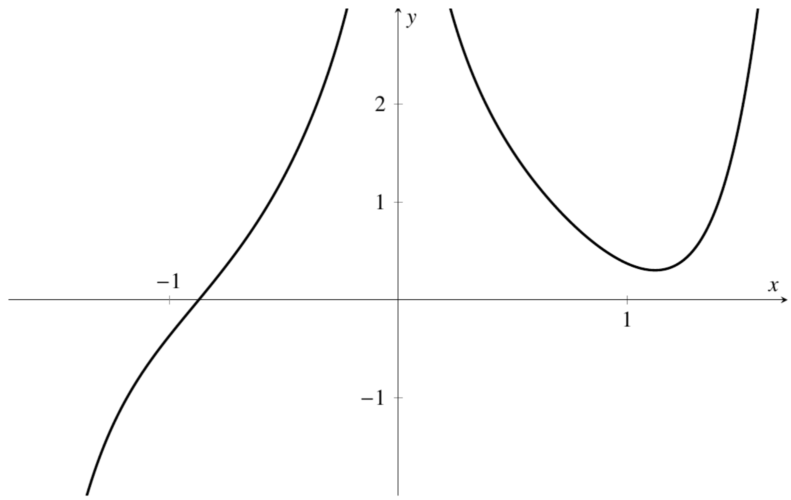
C 1

D $\sqrt{3}$

Question 15

The graph of a curve C is given below for certain values of x.

What could be the equation of C?



A $y = x^3 e^{\cos 3x} + \ln x^2$

B $y = x^3 e^{\sin 3x} + \ln x^2$

C $y = x^3 e^{\cos 4x} + \ln x^2$

D $y = x^3 e^{\sin 4x} + \ln x^2$

E $y = x^3 e^{\cos 3x} - \ln x^2$

F $y = x^3 e^{\sin 3x} - \ln x^2$

G $y = x^3 e^{\cos 4x} - \ln x^2$

H $y = x^3 e^{\sin 4x} - \ln x^2$

Question 16

Which of the following is a **necessary but not sufficient** condition for p, such that the curves $y = x^3 - x$ and $y = 2x + p$ intersect exactly once?

A $|p| > 1$

B $p > 2$

C $|p| > 2$

D $p = 3$

E $|p| > 3$

Question 17

The following theorem is quoted during a lecture about mathematical analysis:

Weierstrass M-test

Let $f_1, f_2, ...$ be a sequence of functions $f_i : X \to \mathbb{R}$. If there are constants M_k for all $k \geq 1$ such that $|f_k(x)| \leq M_k$ for all $x \in X$ and $\sum_{k=1}^{\infty} M_k$ converges, then the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely and uniformly on X.

(It is not required to know the concepts of absolute and uniform convergence in this question.)

What is the negation of the above theorem?

A There do not exist constants M_k for any $k \geq 1$ such that $|f_k(x)| \leq M_k$ for any $x \in X$ and $\sum_{k=1}^{\infty} M_k$ converges.

B There exist constants M_k for any $k \geq 1$ such that $|f_k(x)| \leq M_k$ for any $x \in X$, $\sum_{k=1}^{\infty} M_k$ converges, and $\sum_{k=1}^{\infty} f_k(x)$ does not converge absolutely or uniformly on X.

C There exists a constant M_k for some $k \geq 1$ such that $|f_k(x)| \leq M_k$ for any $x \in X$, $\sum_{k=1}^{\infty} M_k$ converges, and $\sum_{k=1}^{\infty} f_k(x)$ does not converge absolutely and uniformly on X.

D There exist constants M_k for any $k \geq 1$ such that $|f_k(x)| \leq M_k$ for any $x \in X$, and $\sum_{k=1}^{\infty} M_k$ converges, and $\sum_{k=1}^{\infty} f_k(x)$ does not converge absolutely and uniformly on X.

Question 18

A student is to prove the following theorem:

For any positive real numbers $a, b, c \in \mathbb{R}$, if $a^2 + b^2 + c^2 = 1$ then $a + b + c \leq \sqrt{3}$.

The student uses some of the statements below to construct a direct deductive proof of the above theorem. They are not necessarily arranged in the correct order.

I. Substitute $a^2 + b^2 + c^2 = 1$ into the inequality to get $(a + b + c)^2 \leq 3$.

II. Assume $a + b + c \leq \sqrt{3}$. Since $a, b, c > 0$, squaring both sides gives $(a + b + c)^2 \leq 3$.

III. We have $ab + bc + ca \leq a^2 + b^2 + c^2$.

IV. We have $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$.

V. Since $a, b, c > 0$, have $a + b + c > 0$. Take the positive square root to get $a + b + c \leq \sqrt{3}$.

VI. For all real numbers a, b, c, have $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$.

VII. Therefore, $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$, proving the theorem.

What is the correct flow of statements that the student needs?

A VI $\rightarrow$ III $\rightarrow$ IV $\rightarrow$ I $\rightarrow$ V

B VI $\rightarrow$ IV $\rightarrow$ III $\rightarrow$ I $\rightarrow$ V

C II $\rightarrow$ I $\rightarrow$ IV $\rightarrow$ III $\rightarrow$ VI

D II $\rightarrow$ VII $\rightarrow$ VI $\rightarrow$ III $\rightarrow$ IV

E I $\rightarrow$ V $\rightarrow$ VI $\rightarrow$ III $\rightarrow$ IV

Question 19

Consider the following theorem:

Let f be a function on the real numbers whose first and second derivatives exist. If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

The following statements, I, II and III, are made.

I. $f'(x) > 0$ is a **necessary** condition for f to be strictly increasing.

II. $f(x)$ converges to a finite real number as $x \to \infty$ **only if** $f'(x)$ converges to zero as $x \to \infty$.

III. $\displaystyle\int_a^b f(x)\,dx < \frac{1}{2}(b-a)\left(f(a)+f(b)\right)$ is a **sufficient** condition for $f''(x) > 0$ for all x such that $a < x < b$.

Which of the above is/are correct?

A None of the above

B I

C II

D III

E I and II

F II and III

G I and III

H I, II and III

Question 20

Consider the following proof that everyone on Earth has the same name.

I. Let $P(n)$ be the statement that every group of n people has the same name. Then $P(1)$ is true since every person in a group of one has the same name as every person in that group.

II. Assume that for every group of k people, everyone has the same name as everyone else in that group.

III. Consider an arbitrary group of $k+1$ people $p_1, \ldots, p_{k+1}$.

IV. By the assumption, the people $p_1, \ldots, p_k$ have the same name since they form a group of k people.

V. Similarly, the people $p_2, \ldots, p_{k+1}$ have the same name since they also form a group of k people.

VI. Hence the people $p_1, \ldots, p_{k+1}$ have the same name. But $p_1, \ldots, p_{k+1}$ can be anyone since they come from an arbitrary group, thus $P(k+1)$ is true.

VII. Since $P(1)$ is true and $P(k)$ implies $P(k+1)$ for all integers $k \geq 1$, every group of n people has the same name.

Where is the first error in the proof?

A Line I

B Line II

C Line III

D Line IV

E Line V

F Line VI

G Line VII