TMUA

COMMUNITY PAPERS

Nusret Balcı TMUA Paper 1

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: Nusret Balcı

Question 1

Evaluate the integral

$$\int_0^1 x^5(x+2)^2(3x+4)\,\mathrm{d}x.$$

A 9

B 11

C $\frac{23}{3}$

D 17

E 6

F 8

G 28

H 41

Question 2

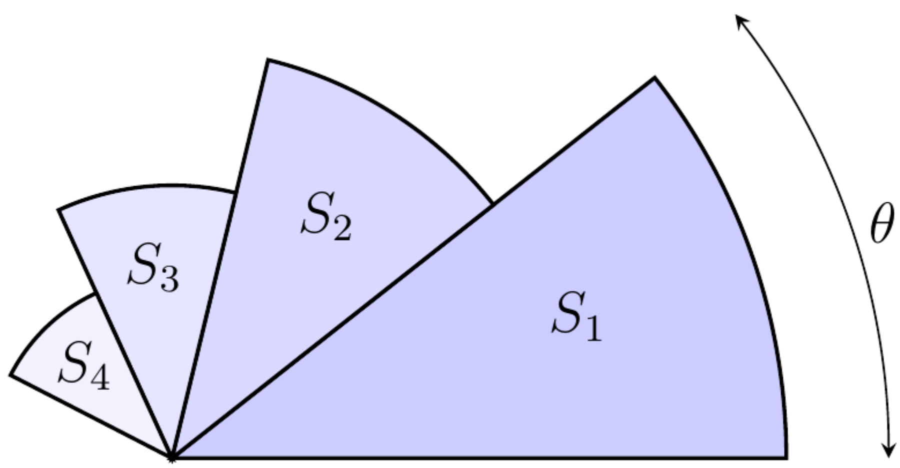
A sequence of similar sectors $S_1, S_2, S_3, ...$ has a constant central angle θ (in radians). The radius r_{n+1} of each sector is defined by the arc length s_n of the previous sector.

Given that the sum of the perimeters of all sectors is exactly three times the perimeter of the first sector:

$$\sum_{n=1}^{\infty} \text{Perimeter}(S_n) = 3 \times \text{Perimeter}(S_1)$$

Find the ratio of the total area of all sectors to the area of the first sector:

$$\text{Ratio} = \frac{\sum_{n=1}^{\infty} \text{Area}(S_n)}{\text{Area}(S_1)}$$



A 2.6

B 3

C 1.1

D 1.9

E 1.8

F 4.3

G $\sqrt{2}$

H $\sqrt{3}$

Question 3

A geometric sequence has first term a and common ratio r, where a and r are positive integers and $r > 1$.

Define the *alternating partial sums*

$$A_n = a - ar + ar^2 - ar^3 + \cdots + (-1)^{n-1}ar^{n-1}.$$

It is given that

$$A_{20} = kA_{10}$$

for some positive integer k.

What is the *smallest possible value* of k?

A 2^{10}

B $2^{10} - 1$

C $2^{10} + 1$

D $2^{20} + 1$

E $\frac{2^{10}+1}{2^{10}-1}$

F $3^{10} + 1$

Question 4

Find the number of solutions to the equation

$$5 \sec 6x = 3x, \qquad -\pi \leq x \leq \pi.$$

A 0

B 1

C 2

D 3

E 4

F 5

G 6

H 7

Question 5

The function $f(x)$ is defined for all real numbers x by:

$$f(x) = 8^x - 6 \cdot 4^x + 9 \cdot 2^x$$

Let S be the set of all integer values of k such that the equation $f(x) = k$ has exactly three distinct real solutions.

What is the number of elements in the set S?

A 0

B 1

C 2

D 3

E 4

F 5

G Infinitely many

Question 6

Let $k \in \mathbb{R}$. Consider the equation

$$(1 + 3\cos\theta)^2 = k\cos 2\theta, \qquad 0^\circ \leq \theta \leq 360^\circ.$$

For which values of k does the equation have *exactly four distinct solutions* in the given interval?

A $k < -9$

B $-9 < k < -1$

C $-1 < k < 0$

D $0 < k < 1$

E $1 < k < 9$

F $k < 0$ or $k > \frac{7}{2}$

G $k < 0$ or $k > 4$

H $k < 0$ or $k > 16$

Question 7

It is given that $f(x) = x^2 - 4x$.

Two curves are defined by:

$$C_1 : y = f(kx)$$
$$C_2 : y = f(x - p)$$

where k and p are constants such that $k > 1$ and $p > 0$.

The curves C_1 and C_2 intersect at two distinct points, A and B. It is given that the midpoint of the line segment AB lies on the y-axis.

Which of the following expresses k in terms of p?

A $k = \frac{p+2}{2}$

B $k = \frac{p+4}{4}$

C $k = \frac{p+4}{2}$

D $k = \frac{p+2}{p}$

E $k = \frac{p-2}{2}$

F $k = \sqrt{p+1}$

Question 8

Point P lies on the circle with equation:

$$(x-1)^2 + (y-2)^2 = 4$$

Point Q lies on the circle with equation:

$$(x-k)^2 + (y-14)^2 = 9$$

where k is a constant and $k > 0$.

The maximum possible length of the line segment PQ is 18.

What is the value of k?

A 4

B 5

C 6

D 8

E 13

F 14

G $11 + \sqrt{13}$

Question 9

The difference between the maximum and minimum values of the function

$$f(x) = a^{\sin x} - a^{-\sin x}$$

where $a > 0$ and x is real, is 3.

Find the sum of the possible values of a.

A 2

B $\frac{5}{2}$

C 3

D $\sqrt{13}$

E $2\sqrt{2}$

F 4

G 5

Question 10

The points $A(2,3)$ and $B(9,-1)$ are two vertices of a triangle. The third vertex lies on the line $y = x$ and has coordinates $C(k,k)$.

Given that the triangle ABC is right-angled, find the sum of all possible values of k.

A $\frac{47}{2}$

B 24

C $\frac{53}{2}$

D 28

E $\frac{59}{2}$

F 31

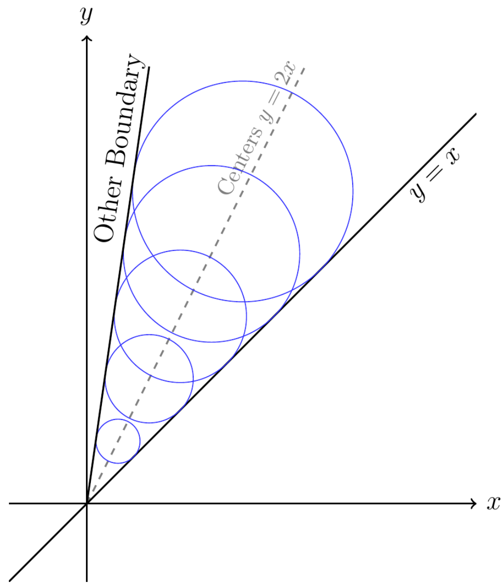
G 33

H 37

Question 11

A family of circles is defined such that their centers lie on the line $y = 2x$. Every circle in the family is tangent to the line $y = x$.

The union of all such circles forms a region R in the xy-plane bounded by two straight lines intersecting at the origin. One boundary is the line $y = x$. Find the equation of the other boundary line.



A $y = 3x$

B $y = 4x$

C $y = 5x$

D $y = 6x$

E $y = 7x$

F $y = 8x$

G $y = 9x$

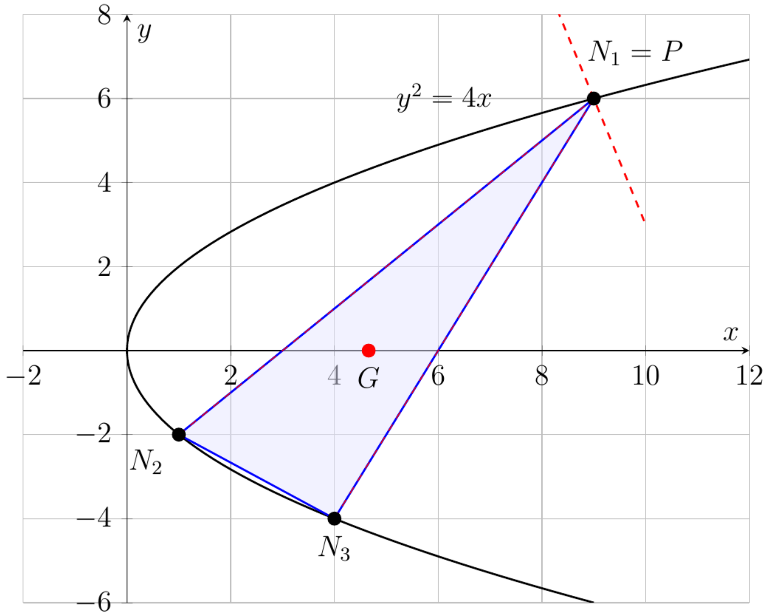
H $y = 11x$

Question 12

The point $P(x_P, y_P)$ lies on the curve C with equation $y^2 = 4x$, such that $y_P > 0$.

Normals are drawn to the curve C at three distinct points, N_1, N_2, and N_3. All three normals pass through the point P. Note that one of these points is P itself (i.e., let $N_1 = P$).

Let G be the centroid of the triangle formed by the points N_1, N_2, and N_3. Find the set of all possible x-coordinates for G.



A $x > -\frac{4}{3}$

B $x > 0$

C $x > \frac{8}{3}$

D $x > 4$

E $x > \frac{16}{3}$

F $x > 8$

G $x < 3$

H $1 < x < 4$

Question 13

Consider the polynomial

$$f(x) = x^3 - 3ax + b$$

where a and b are real parameters.

The parameter a varies in the range $0 \leq a \leq 1$.

The parameter b varies in the range $-2 \leq b \leq 2$.

Assuming that the pair (a, b) is chosen uniformly at random from the rectangle defined by these ranges, find the probability that the equation $f(x) = 0$ has three distinct real roots.

A $\frac{1}{5}$

B $\frac{2}{5}$

C $\frac{3}{10}$

D $\frac{4}{15}$

E $\frac{1}{3}$

F $\frac{1}{4}$

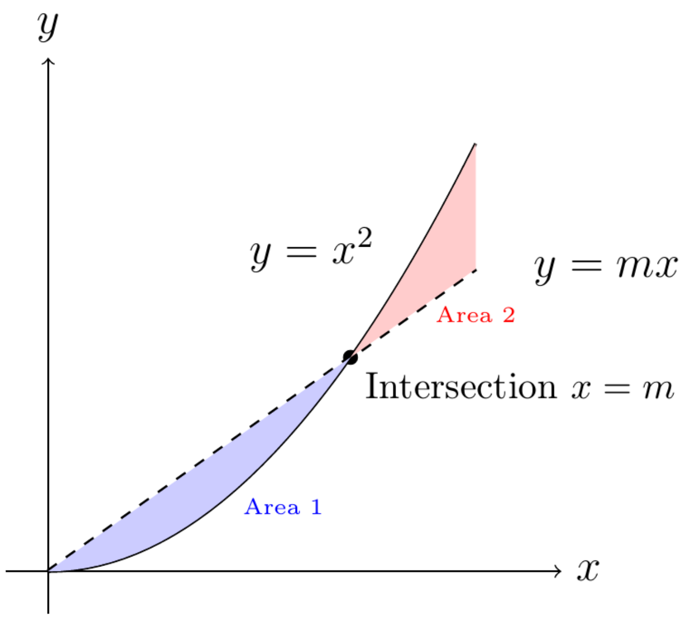
G $\frac{8}{25}$

Question 14

Let m be a real constant such that $0 \leq m \leq 1$. We wish to approximate the curve $y = x^2$ with the line $y = mx$ on the interval $[0, 1]$. Let $E(m)$ be the total area of the region bounded between the curve $y = x^2$ and the line $y = mx$, for $0 \leq x \leq 1$.

$$E(m) = \int_0^1 \left|x^2 - mx\right| \, \mathrm{d}x$$

Find the value of m for which $E(m)$ is a minimum.



A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{1}{\sqrt{2}}$

D $\frac{1}{\sqrt{3}}$

E $\frac{3}{4}$

F $\frac{\sqrt{3}}{2}$

Question 15

Let p, q, r, s be real numbers with $p < q < r < s$. Consider the function $f(x)$ defined by:

$$f(x) = |x - p| + |x - q| + |x - r| + |x - s|$$

The set of values of x for which $f(x)$ takes its minimum value is the closed interval $[5, 12]$. The minimum value of $f(x)$ is 28.

What is the value of $s - p$?

A 7

B 14

C 19

D 21

E 28

F 35

G It cannot be determined from the given information.

Question 16

The function $f(x)$ is a polynomial that satisfies the equation

$$f'(x) + \int_0^x f(t)\,\mathrm{d}t = x^3 + kx^2 + x + 2$$

for all real values of x, where k is a real constant.

What is the value of k?

A 1

B 2

C 3

D 4

E No such value of k exists.

Question 17

A sequence of real numbers $v_0, v_1, v_2, \ldots$ is defined by $v_0 = 0$, $v_1 = 1$, and the linear recurrence relation:

$$v_{n+1} = \sqrt{2}\, v_n - v_{n-1} \qquad \text{for } n \geq 1.$$

What is the value of v_{101}?

A -1

B 0

C 1

D $-\sqrt{2}$

E $\sqrt{2}$

F $1 - \sqrt{2}$

G $\sqrt{2} - 1$

H $1 + \sqrt{2}$

Question 18

Let $f(x)$ and $g(x)$ be two polynomials with coefficients in $\mathbb{R}$, defined as:

$$f(x) = x^2 - 8x + \alpha$$
$$g(x) = -x^2 + 4x + \beta$$

When both $f(x)$ and $g(x)$ are divided by $x - k$, the remainders are 5 and 7, respectively.

Given that $k \in \mathbb{R}$, find the largest possible value for the quantity $\alpha - \beta$.

A 12

B 13

C 14

D 15

E 16

F 18

G 20

H 22

Question 19

Consider the following inequality:

$$m|x| + 1 \leq (x-3)^2 + 8$$

where m is a real constant.

Which one of the following describes the complete set of values of m such that the inequality is true for all real x?

A $m \leq \frac{7}{3}$

B $m \leq 2$

C $m \leq \frac{5}{3}$

D $m \leq \frac{3}{2}$

E $m \leq \frac{4}{3}$

F $m \leq 1$

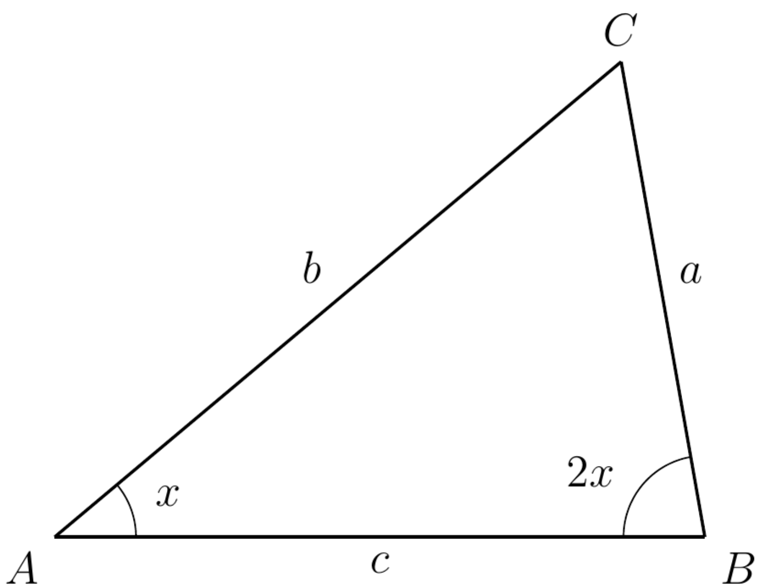
G $m \leq \frac{1}{2}$

H There are no such values of m

Question 20

Consider a non-degenerate triangle with angles given by x and $2x$ (measured in degrees). The side lengths of this triangle form a non-constant Geometric Progression (in some specific order).

Which of the following options correctly describes the number and location of all possible values for x?



A There are no such values of x; such a triangle cannot exist.

B There is exactly one value of x, and it lies in the interval $(30^\circ, 36^\circ)$.

C There is exactly one value of x, and it lies in the interval $(36^\circ, 45^\circ)$.

D There is exactly one value of x, and it lies in the interval $(45^\circ, 60^\circ)$.

E There are exactly two values of x: one in $(30^\circ, 36^\circ)$ and one in $(45^\circ, 60^\circ)$.

F There are exactly two values of x: one in $(36^\circ, 45^\circ)$ and one in $(45^\circ, 60^\circ)$.

G There are exactly two values of x: both lie in the interval $(36^\circ, 45^\circ)$.

H There are exactly three values of x: one in $(30^\circ, 36^\circ)$, one in $(36^\circ, 45^\circ)$, and one in $(45^\circ, 60^\circ)$.