TMUA

COMMUNITY PAPERS

TMUAmockz Set 1 Paper 2

Recommended time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

This community paper is presented in a clean, print-ready format. The wording, mathematics, answer choices, and diagrams are preserved from the source paper.

Paper guidance

- Attempt every question. Select the best answer from the choices given.
- If a diagram is shown, it is reproduced from the source material and may not be drawn to scale.
- This question paper does not include worked solutions or a marked answer key.

Best of luck, and enjoy the challenge!

Source credit: TMUAmockz

Question 1

Let p and q be real constants with $p \neq q$. Two curves are defined by

$$C_1 : y = (x-p)^2(x-q) \quad \text{and} \quad C_2 : y = (x-q)^2(x-p)$$

Consider the following four statements.

I. The curves C_1 and C_2 always intersect at exactly two distinct points.

II. The curve C_1 always has a local minimum at $x = p$.

III. The sum of the y-coordinates of all intersection points of C_1 and C_2 is always zero.

IV. The tangent to C_1 at the point where $x = p$ is also tangent to C_2 at that point.

Which of the following statements are always true?

A I and III only

B II and IV only

C I and IV only

D I, III and IV only

E II, III and IV only

F I, II, III and IV

Question 2

The distinct points P and Q with coordinates $(p^2, 2p)$ and $(q^2, 2q)$ respectively lie on the curve C with equation $y^2 = 4x$. Consider the following two statements:

S: The chord PQ passes through the point $(-1, 0)$

T: $p^2q^2 = 1$

Which of the following correctly describes the relationship between S and T?

A S is not necessary and not sufficient for T

B S is necessary but not sufficient for T

C S is sufficient but not necessary for T

D S is necessary and sufficient for T

E T is sufficient but not necessary for S

F S is necessary and sufficient for T, but only when $p, q > 0$

Question 3

The equation $ax^2 + bx + c = 0$, where a, b, c are real and $a \neq 0$, is said to have property R if it has two distinct real roots. Consider the following four statements.

I. $b^2 - 4ac > 0$ is necessary and sufficient for property R.

II. a and c having opposite signs is a sufficient condition for property R.

III. a and c having opposite signs is a necessary condition for property R.

IV. $b \neq 0$ is a necessary condition for property R.

A I only

B I and II only

C I and III only

D I, II and III only

E I, II, III and IV

F II and IV only

Question 4

A data set consists of seven values:

$$4, \quad 1, \quad 15, \quad x, \quad 1, \quad 10, \quad 1$$

where x is a real number. The mean, median and mode of this data set form a non-constant arithmetic progression. Find the sum of all possible values of x.

A 3

B 14

C 17

D 18

E 20

F 21

Question 5

Real numbers p, q, r satisfy $p < q < r$, and n is a positive integer. Which of the following statements must be true?

I. $p^n q^n < q^{2n}$ only if n is even

II. $p^2 + r^2 > 2q^2$ only if $q < 0$

III. $(r - p)^2 > (r - q)^2 + (q - p)^2$

A None of them

B I only

C II only

D III only

E I and III only

F II and III only

Question 6

Let x be an integer. Consider the expression

$$\frac{5x}{12} + \frac{x}{8}.$$

Which of the following statements about x are true whenever this expression is an integer?

I x being a multiple of 24 is both necessary and sufficient for the expression to be an integer.

II x being a multiple of 12 is sufficient for the expression to be an integer.

III x must be a multiple of 8.

IV x^2 must be a multiple of 576.

A I and III only

B I and IV only

C I, III and IV only

D I, II, III and IV

E II, III and IV only

F III and IV only

Question 7

A line ℓ with gradient m is tangent to the curve $y = x^2$ and passes through the point $(10, 36)$. Which of the following is a necessary and sufficient condition on m for ℓ to be tangent to $y = x^2$?

A $m = 20$

B $m^2 - 40m + 36 = 0$

C $m^2 - 40m + 144 = 0$

D $m^2 - 20m + 36 = 0$

E $m = 40$

F $m^2 - 20m + 144 = 0$

Question 8

Let p and q be distinct positive real numbers, and define $I = \int_0^1 x^p \, \mathrm{d}x$ and $J = \int_0^1 x^{1/q} \, \mathrm{d}x$.

Which of the following is a necessary and sufficient condition for $I > J$?

A $p < q$

B $p + q < 2$

C $pq < 1$

D $p < 1$ and $q > 1$

E $pq > 1$

F $p < q < 1$

Question 9

Let a, b be real constants. Let n be the number of real solutions to the equation $x^2 - a|x| + b = 0$. Which of the following statements are true?

I. $n = 4$ if and only if $a^2 > 4b > 0$ and $a > 0$

II. $n = 2$ only if $b \geq 0$

III. $n = 0$ if $4b > a^2$

A None of them

B I only

C II and III only

D III only

E I and III only

F I, II and III

Question 10

Given that $f''(x) > 0$ for all x with $0 \leq x \leq 1$, which of the following statements must be true?

I. $f\left(\frac{1}{2}\right) < \frac{f(0)+f(1)}{2}$

II. $\int_0^1 f(x)\,\mathrm{d}x < f\left(\frac{1}{2}\right)$

III. $f'(0) < f(1) - f(0)$

A I only

B II only

C III only

D I and II only

E I and III only

F I, II and III

Question 11

The first term of an infinite geometric progression is $\cos x$ and the common ratio is also $\cos x$. For how many values of x in the interval $0 \leq x < 2\pi$ does the sum to infinity equal $\sin x$?

A 0

B 2

C 3

D 4

E 6

F 1

Question 12

A student claims that $\log_3(x-4)+\log_3(22-x) < 4$ holds for all integers x with $4 < x < 22$. Is the student correct? If not, state the integer values for which the inequality fails and find the total number of integers with $4 < x < 22$ for which it does hold.

A The student is correct; all 17 integers satisfy the inequality

B The student is incorrect; the inequality fails for $x = 4$ and $x = 22$ only

C The student is incorrect; the inequality fails for 2 integers

D The student is incorrect; the inequality fails for exactly 1 integer, giving 16 valid values

E The student is incorrect; the inequality fails for 3 integers

F The student is incorrect; the inequality fails for 4 integers

Question 13

An ant sits on the surface of a regular tetrahedron with edges of length 1. It crawls along the surface from the midpoint of one edge to the midpoint of the opposite edge by the shortest possible route. (Two edges are opposite if they share no common vertex.)

What is the length of the shortest such route?

A $\frac{\sqrt{3}}{2}$

B 1

C $\sqrt{2}$

D $\frac{3}{2}$

E $\sqrt{3}$

F 2

Question 14

The function $f(x) = x^3 + ax^2 - a^2x + b$ has a local maximum at $x = a/3$ and three distinct real roots. What conditions must a and b satisfy?

A $a < 0$ and $-\frac{5a^3}{27} < b < -a^3$

B $a > 0$ and $\frac{5a^3}{27} < b < -a^3$

C $a < 0$ and $-a^3 < b < \frac{5a^3}{27}$

D $a < 0$ and $\frac{5a^3}{27} < b < -a^3$

E $a < 0$ and $\frac{4a^3}{27} < b < -a^3$

F $a < 0$ and $\frac{5a^3}{27} < b < a^3$

Question 15

A curve C has equation $y = px^2 + qx + r$, where $p \neq 0$. The curve C is reflected in the x-axis to give a curve C'. The curve C is then translated 3 units to the right to give the graph of $y = f(x)$, and C' is translated 3 units to the left to give the graph of $y = g(x)$.

Consider the following four statements about the graph of $y = (f + g)(x)$.

I The graph of $y = (f + g)(x)$ is a straight line.

II The y-intercept of the line equals $-r$.

III The gradient of the line is determined solely by p; it does not depend on q or r.

IV If $p > 0$ and $q > 0$, the line has negative gradient and negative y-intercept.

A I and II only

B III and IV only

C I, III and IV only

D II and III only

E I, II and III

F II, III and IV only

Question 16

The expansion of $(x^a + x^{-b})^n$, where a, b, n are positive integers, contains a constant term. Which of the following is/are necessary and sufficient conditions for this, for some non-negative integer k?

I. $k(a+b) = an$

II. $k(a+b) = bn$

III. $ka = nb$

A I only

B II only

C III only

D I and II only

E I and III only

F II and III only

Question 17

Two parabolas have equations $f(x) = x^2 + px + q$ and $g(x) = x^2 + rx + s$, where p, q, r and s are integers each chosen independently and uniformly at random from 1 to 6 inclusive.

Consider the following four statements.

I $p \neq r$ is necessary and sufficient for the parabolas to intersect at exactly one point.

II The probability that the parabolas have at least one point in common is $\frac{31}{36}$.

III The probability that the two parabolas are identical is $\frac{1}{36}$.

IV If $p = r$, then the two parabolas have the same vertex.

A I and II only

B II and III only

C I, II and III only

D I, II, III and IV

E I and III only

F II, III and IV only

Question 18

Three circles with radii p, q and r, where $p < q < r$, are mutually externally tangent. Let P, Q, R denote their respective centres. Which of the following is a necessary and sufficient condition for the triangle PQR to have a right angle at P?

A $p + q = 2r$

B $q^2 = pr$

C $(p + q)(p + r) = 2qr$

D $p(q + r) = 2qr$

E $(p + q + r)^2 = 2qr$

F $p(p + q + r) = 2qr$

Question 19

Given that p and q are real constants, which of the following is a necessary but not sufficient condition for the equation

$$qx + 2p = x^3 + qx^2 + (p - q)x - 8$$

to have exactly one real solution?

A $p > 3$

B $q \leq -2$

C $4p \geq q^2 + 4q - 12$

D $p > q + 3$

E $4p > q^2 + 4q - 12$

F $p + q > 1$

Question 20

A sequence a_n is defined by $a_{n+1} = \frac{a_n+1}{1-a_n}$ and it is given that $a_{13} = \sqrt{3}$. Find the value of $\sum_{n=0}^{30} a_n$.

A 0

B $-\frac{28\sqrt{3}}{3}$

C $-4\sqrt{3}$

D $-\frac{31\sqrt{3}}{3}$

E $-31\sqrt{3}$

F $-\frac{4\sqrt{3}}{3}$